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Theoretical analysis of integer programming models for the two-dimensional two-staged knapsack problem

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Abstract

In this study, we theoretically compare integer programming models for the two-dimensional two-staged knapsack problem. Including the well-known level packing model, we introduce two pattern-based models called the strip packing model and the staged pattern model derived from integer programming models for the two-dimensional two-staged cutting stock problem. We show that the level packing model provides weaker linear programming (LP) relaxation bounds than pattern-based models. Furthermore, we also present upper bounds on the LP-relaxation bound of the level packing model, which can be obtained from the LP-relaxation bounds of the pattern-based models.

Keywords Two-dimensional two-staged knapsack problem · Integer programming models · Level packing model · Strip packing model · Staged pattern model · LP-relaxation

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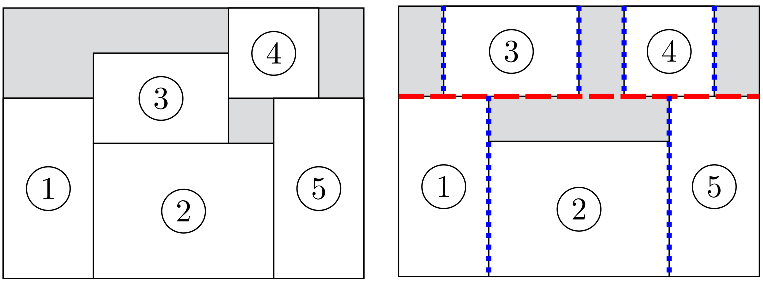
1 Introduction

The two-dimensional two-staged knapsack problem (2DK in short) produces a set of small rectangular items by cutting a large rectangular plate. Formally, we are given a large rectangular plate S with height H and width W, and a list of m types of small rectangular items to be cut from S. An item of type i $(i = 1, \dots, m)$ has a height $h_i \leq H$, a width $w_i \leq W$, a profit p_i, and an upper bound (demand) d_i which denotes the maximum number of items of type i allowed to be cut. Furthermore, we do not allow rotations of items. The objective is to maximize the total profit obtained from the set of cut items. This problem is closely related to the two-dimensional two-staged cutting stock problem (2DCS in short) [4]. The 2DCS minimizes the number of plates required to cut the full demand of items, unlike the 2DK.

The 2DK is a variant of the two-dimensional knapsack problem [5], which conventionally assumes that the items have to be cut in the orthogonal style: the vertical and horizontal sides of each item must be parallel to the vertical and horizontal sides of the large plate, respectively. In many practical application, additional constraints are imposed to the 2DK. We consider one of the common additional constraints, which is that the cuts should be guillotine type, i.e., each cut must divide the plate into two rectangles. Each item in the 2DK is obtainable from two-stage guillotine cuts, where the first- and second-stage cuts are orthogonal. Without loss of generality, we assume that vertical cuts are made after all horizontal cuts are carried out. If vertical cuts should be made first, we swap the widths and heights of the plate and items in the given problem instance. In addition, if the direction of the first-stage cuts is free, we can solve the 2DK by solving two 2DKs in which the direction of the first-stage cuts is fixed vertically and horizontally, respectively. Especially, we refer to the separated sub-plates produced after the first-stage cutting as the *strips* or *levels*. In general, 2DK and 2DCS are used to represent the unstaged problem. However, we use these abbreviations for the sake of simplicity when referring to the two-staged problems. For the unstaged guillotine cutting problem and the corresponding models, we refer the readers to Russo et al. [16], Iori et al. [11], Becker et al. [3].

A simple example of a two-staged guillotine cutting for five items is illustrated in Fig. 1. Both Fig. 1a and b are two-dimensional orthogonal cuttings. However,



(a) An infeasible case. (b) A feasible case.

Fig. 1 An example of two-staged two-dimensional guillotine cutting only Fig. 1b is a feasible two-staged two-dimensional guillotine cutting. The first- and second-stage cuts are illustrated as dashed and dotted lines, respectively. Additionally, we consider the inexact case of the 2DK, that is, one can require trimming unnecessary areas after the second stage of guillotine cuts, such as the shaded area in Fig. 1b above the item (2).

The 2DK is presented as the pricing problem of the Dantzig-Wolfe decomposition approach for the 2DCS introduced by Gilmore and Gomory [7]. Therefore, the studies on the 2DK may provide crucial clues to solving the 2DCS and their real-world applications. For this reason, the 2DK has received considerable attention, and several studies have proposed solution approaches for the problem. Gilmore and Gomory [7] considered a special case of the 2DK where demands of items are not given, which is called the unconstrained 2DK [8]. Using a dynamic programming approach, the authors devised a pseudo-polynomial time exact algorithm for the unconstrained 2DK. Based on this result, Hifi [8] extended the algorithm for the unconstrained 2DK allowing the rotation of items. Unlike the unconstrained case, the 2DK is referred to as "constrained" when the demands are considered. The following section will provide the formal definition of the constrained and unconstrained 2DK. For the constrained 2DK, Hifi and Roucairol [10] proposed an exact solution approach based on the branch-and-bound algorithm utilizing the bounds computed by dynamic programming techniques. Lodi and Monaci [13] proposed an integer programming model for the constrained 2DK, which can be solved using the standard branch-and-bound algorithm. The authors introduced potential strips (levels) that can be packed into the plate, where each height corresponds to the height of a specific item. The proposed integer programming model, which we call the *level packing model*, determines the usage of each potential level while packing items into each level. The level packing model considers items with the same shape as distinct ones with unit demands. The authors also proposed a variant of the level packing model, which treats items with the same shape as one item with demand larger than 1. The authors proved that the LP-relaxation bounds of the level packing model and its variant are equivalent. In our paper, we only consider the level packing model, not the variant. Subsequently, the authors derived another integer programming model for the constrained 2DK based on the concept of "width patterns" proposed by Gilmore and Gomory [7], each of which represents a set of item widths that can be cut from the large plate along the horizontal side. We refer to this pattern-based model for the constrained 2DK as the *strip packing model*. Belov and Scheithauer [4] proposed a branch-and-cut-and-price algorithm using Chvátal-Gomory and Gomory mixed-integer cuts for the strip packing model. Heuristic algorithms for the 2DK have also been devised as part of efficient solution approaches for the 2DCS; see Hifi and Roucairol [10], Hifi and M'Hallah [9], and Alvarez-Valdes et al. [2].

On the other hand, various integer programming models for the 2DCS have also been devised in the literature. Macedo et al. [14] extended to the 2DCS the arc-flow formulation, proposed by Valério de Carvalho [19] for the one-dimensional cutting stock problem. Mrad et al. [15] utilized the "height patterns", each of which represents a set of item heights that can be cut from the large plate along the vertical side. Subsequently, the authors proposed another pattern-based model for the 2DCS, which utilizes both width and height patterns. Moreover, Silva et al. [17]

proposed an extension to the 2DCS of the one-cut model, as originally proposed by Dyckhoff [6] for the one-dimensional cutting stock problem. The relationship between these models for the 2DCS has also been well-established. Kwon et al. [12] analyzed the theoretical hierarchy between the bounds provided by the linear programming (LP) relaxations of these models, along with comprehensive computational comparisons.

The studies on the 2DCS can be adapted to the 2DK due to their relevance. In particular, the integer programming models for the 2DCS can be utilized to formulate the 2DK and develop efficient solution approaches. However, despite the extensive studies on models for the 2DCS, to the best of our knowledge, only the level packing model (with its variant) and the strip packing model were proposed for the constrained 2DK in Lodi and Monaci [13]. Furthermore, although Lodi and Monaci [13] performed computational comparisons of the bounds obtained from the LP-relaxations of the level packing model and the strip packing model, the theoretical comparison between them has hardly been addressed.

In this study, we discuss three integer programming models for the constrained 2DK: an extension of the level packing model, the strip packing model, and another pattern-based model adapted from the model for the 2DCS proposed by Mrad et al. [15]. Subsequently, we conduct a theoretical comparison of the LP-relaxations of these models. Our contributions can be summarized as follows:

- We modify the level packing model of Lodi and Monaci [13] by adding a set of valid inequalities, which enhance the LP-relaxation bound.
- By utilizing the 2DCS model introduced by Mrad et al. [15], we present another pattern-based model for the constrained 2DK, which we refer to as the *staged pattern model.*
- We establish a theoretical relationship among the LP-relaxation bounds of the three models under consideration: the modified level packing model, the strip packing model inherited from Gilmore and Gomory [7], and the staged pattern model adapted from Mrad et al. [15]. Our analysis shows that the pattern-based models, i.e., the strip packing model and the staged pattern model, yield tighter LP-relaxation bounds compared to the level packing model. Moreover, we derive upper bounds on the level packing model by utilizing the LP-relaxation bounds of the pattern-based models. We also provide a concrete example illustrating the tightness of these upper bounds.

The remainder of this paper is organized as follows. In Sect. 2, we provide a formal definition of the 2DK, and introduce the integer programming models under consideration for the problem. These models are compared theoretically with respect to the bounds obtained from their LP-relaxations in Sect. 3. A tight example for the comparison is also given in Sect. 3. Finally, concluding remarks are given in Sect. 4.

2 Integer programming models for the 2DK

This section presents three integer programming models for the 2DK: the level packing model, the strip packing model, and the staged pattern model. Before starting, we provide some formal definitions concerned with the 2DK.

An instance of the 2DK can be defined with input $(m, H, W, \boldsymbol{h}, \boldsymbol{w}, \boldsymbol{d}, \boldsymbol{p})$ where $\boldsymbol{h} = (h_1, \dots, h_m)$, $\boldsymbol{w} = (w_1, \dots, w_m)$, $\boldsymbol{d} = (d_1, \dots, d_m)$ and $\boldsymbol{p} = (p_1, \dots, p_m)$. Without loss of generality, we assume that all the values in the tuples are integers and $h_1 \geq h_2 \geq \cdots \geq h_m$. Furthermore, when $d_i > \lfloor H/h_i \rfloor \lfloor W/w_i \rfloor$ for some $i \in M$ where $M = \{1, \dots, m\}$, we treat the demand as $d_i = \lfloor H/h_i \rfloor \lfloor W/w_i \rfloor$ by taking the geometric limit into consideration. The 2DK is called constrained when $d_i < \lfloor H/h_i \rfloor \lfloor W/w_i \rfloor$ for some $i \in M$. On the contrary, if $d_i = \lfloor H/h_i \rfloor \lfloor W/w_i \rfloor$ for all $i \in M$, we call the 2DK unconstrained. We mainly focus on the constrained 2DK in this study, however, the results remain valid for the unconstrained 2DK.

As described in Sect. 1, the two-staged guillotine cuttings are applied to a given large plate in the 2DK. The first-stage cutting produces strips that are sub-plates separated from the given plate. Each strip is separated into items by the second-stage cutting, with trimming if needed. Among the items separated from each strip, the item with the largest height is referred to as a *strip defining item*. Of course, the height of each strip is equivalent to that of the strip defining item.

Using these definitions, we first formally describe the level packing model proposed by Lodi and Monaci [13] in the following discussion. Subsequently, we enhance the level packing model by introducing its valid inequalities. Additionally, two pattern-based models are discussed, the strip packing model that uses only width patterns and the staged pattern model that uses also height patterns.

2.1 Level packing model

A given 2DK instance $(m, H, W, \boldsymbol{h}, \boldsymbol{w}, \boldsymbol{d}, \boldsymbol{p})$ can be equivalently transformed into another one by regarding d_i items of the type i as d_i distinct item types with a unit demand for each $i \in M$. Let $N = \{1, \dots, n\}$ be the set of item types of the transformed instance, where $n = \sum_{i=1}^{m} d_i$. We define $\beta_j = \min\{i \in M : \sum_{k=1}^{i} d_k \geq j\}$ for each $j \in N$, which indicates the item type of the original instance corresponding to the type $j \in N$ of the transformed instance. Let $\bar{h}_j = h_{\beta_j}$, $\bar{w}_j = w_{\beta_j}$, and $\bar{p}_j = p_{\beta_j}$ for each $j \in N$. Then, an optimal solution of the 2DK instance $(m, H, W, \boldsymbol{h}, \boldsymbol{w}, \boldsymbol{d}, \boldsymbol{p})$ can be obtained by solving $(n, H, W, \bar{\boldsymbol{h}}, \bar{\boldsymbol{w}}, \mathbb{1}, \bar{\boldsymbol{p}})$ where $\bar{\boldsymbol{h}} = (\bar{h}_1, \dots, \bar{h}_n)$, $\bar{\boldsymbol{w}} = (\bar{w}_1, \dots, \bar{w}_n)$ and $\bar{\boldsymbol{p}} = (\bar{p}_1, \dots, \bar{p}_n)$.

In the transformed 2DK instances, Lodi and Monaci [13] defined n potential strips that have distinct strip defining items characterized by their types, from which the items can be cut. Using the concept of the potential strips, Lodi and Monaci [13] proposed the level packing model (LM) for the 2DK, which is described as follows:

$$\text{LM:} \quad \text{maximize} \quad \sum_{k=1}^{n} \sum_{j=k}^{n} \bar{p}_j x_{jk} \tag{1}$$

$$\text{subject to} \quad \sum_{k=1}^{j} x_{jk} \leq 1, \quad \forall j \in N, \tag{2}$$

$$\sum_{j=k+1}^{n} \bar{w}_j x_{jk} \leq (W - \bar{w}_k) x_{kk}, \quad \forall k \in N, \tag{3}$$

$$\sum_{k=1}^{n} \bar{h}_k x_{kk} \leq H, \tag{4}$$

$$x_{jk} = 0, \quad \forall k \in N,\ \forall j \in \{1, \ldots, k-1\}, \tag{5}$$

$$x_{jk} \in \{0, 1\}, \quad \forall j \in N,\ \forall k \in N \tag{6}$$

A decision variable x_{jk}, for each $j \in N$ and $k \in N$, represents whether the item of type j is to be cut from the potential strip defined by the item of type k. Specifically, in the transformed 2DK instance with the definition of the potential strips, the variable x_{kk} for each $k \in N$ can represent the use of the potential strip defined by the item of type k. Constraints (5) ensure that each potential strips is defined by the item of type k for each $k \in N$. Constraints (2) describe the demand limits. Constraints (3) represent that the total width in each potential strip cannot exceed W. Constraint (4) restricts the total height of the used potential strips to H.

Observe that, from constraints (3), each x_{jk} can be positive only if $x_{kk} = 1$. Hence, the following inequalities

$$x_{jk} \leq x_{kk}, \quad \forall k \in N,\ \forall j \in \{k+1, \ldots, n\} \tag{7}$$

are valid for the feasible solution set of LM. By adding these inequalities to LM, we propose a modified level packing model (ML) described as follows:

$$\begin{aligned} \text{ML}: \quad & \text{maximize} \quad (1) \\ & \text{subject to} \quad (2) - (7). \end{aligned}$$

It is clear that the LP-relaxation bound provided by ML is at least as tight as the LP-relaxation bound obtained from LM. Moreover, constraints (7) make it easier to compare the LP-relaxations between ML and the pattern-based models presented in the following section.

2.2 Pattern-based models

As mentioned in Sect. 1, the 2DK can be formulated utilizing width and height patterns. Formally, a width pattern can be represented as a vector $\boldsymbol{a} \in P_w$ where

$$P_w = \left\{ \boldsymbol{a} \in \mathbb{Z}_+^m \setminus \{\boldsymbol{0}\} \ : \ \sum_{i \in M} w_i a_i \le W; \, a_i \le d_i, \ \forall i \in M \right\}.$$

Each width pattern can define a strip for the 2DK. Then, each component of a width pattern represents the number of each item type in the strip corresponding to the width pattern. We note that the set of width patterns, P_w, is finite and discrete. For a procedure to generate all patterns, we refer to Suliman [18]. Let $\boldsymbol{a}^q = (a_1^q, \dots, a_m^q)$ be an element of P_w for each $q \in Q_w$, where $Q_w = \{1, \dots, |P_w|\}$. We also define $\theta(q)$ as the minimum index in the support of $\boldsymbol{a}^q$ for each $q \in Q_w$, where the support denotes the set of indexes with positive components of $\boldsymbol{a}^q$. We note that the support of $\boldsymbol{a}^q$ is not empty for each $q \in Q_w$ since $\boldsymbol{a}^q \neq \boldsymbol{0}$. Here, $\theta(q)$ means the strip defining item for the width pattern $\boldsymbol{a}^q$ for each $q \in Q_w$. Let $Q_w^i = \{q \in Q_w \ : \ \theta(q) = i\}$ for each $i \in M$. Then, Q_w can be partitioned into Q_w^i's, that is, $Q_w = \cup_{i \in M} Q_w^i$ and $Q_w^i \cap Q_w^j = \emptyset$ for all $i, j \in M$ such that $i \neq j$.

A height pattern can also be represented as a vector $\boldsymbol{b} \in P_h$ where

$$P_h = \left\{ \boldsymbol{b} \in \mathbb{Z}_+^m \setminus \{\boldsymbol{0}\} \ : \ \sum_{i \in M} h_i b_i \le H \right\}.$$

We refer to each height pattern in P_h as $\boldsymbol{b}^r = (b_1^r, \dots, b_m^r)$ for each $r \in Q_h$ where $Q_h = \{1, \dots, |P_h|\}$. Unlike width patterns, the demand limits are neglected in the definition of height patterns, that is, height patterns are unconstrained. These height patterns are used later to describe the staged pattern model together with width patterns. In the staged pattern model, the demand limits can be expressed using only the variables associated with width patterns. Therefore, the demand limits are not necessary in the definition of height patterns.

Figure 2 illustrates how these patterns represent a solution for the 2DK. In this example, the large plate is divided into two strips and one waste fragment (the shaded strip) by two first-stage cuts (dashed lines). Subsequently, the second-stage cuts (dotted lines) divide each strip into items with trimming. The set of items produced from each strip by the second-stage cuts corresponds to a width pattern. In other words, a width pattern can represent second-stage cuts. Now, let us consider the strip defining items for the strips. The set of these items

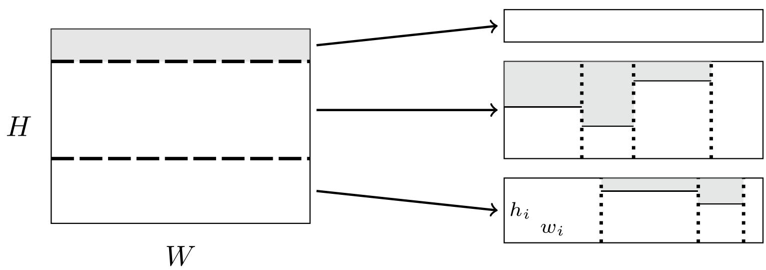


Fig. 2 An illustration of a height pattern and width patterns

corresponds to a height pattern, that is, a height pattern can represent first-stage cuts.

Now, we introduce two pattern-based models for the 2DK. The strip packing model (PM) proposed by Lodi and Monaci [13] utilizes only the width patterns, and it can be formulated as follows:

$$\text{PM:} \quad \text{maximize} \quad \sum_{q \in Q_w} \sum_{i \in M} p_i a_i^q x_q \tag{8}$$

$$\text{subject to} \quad \sum_{q \in Q_w} a_i^q x_q \le d_i, \quad \forall i \in M, \tag{9}$$

$$\sum_{i \in M} \sum_{q \in Q_w^i} h_i x_q \le H, \tag{10}$$

$$x_q \in \mathbb{Z}_+, \quad \forall q \in Q_w. \tag{11}$$

For each $q \in Q_w$, a decision variable x_q represents the number of used strips corresponding to the width pattern $\boldsymbol{a}^q$. Constraints (9) describe the demand limits, and constraint (10) states that the total height of the strips corresponding to the used width patterns cannot exceed H.

An alternative pattern-based model using both width and height patterns can be derived from the 2DCS model introduced in Mrad et al. [15]. We refer to this model as the staged pattern model (SM) for 2DK, which is formulated as follows:

$$\text{SM:} \quad \text{maximize} \quad (8)$$

$$\text{subject to} \quad (9), (11),$$

$$\sum_{r \in Q_h} b_i^r y_r \ge \sum_{q \in Q_w^i} x_q, \quad \forall i \in M, \tag{12}$$

$$\sum_{r \in Q_h} y_r \le 1, \tag{13}$$

$$y_r \in \{0, 1\}, \quad \forall r \in Q_h.$$

Here, the decision variables x_q, the objective function (8), and constraints (9) are the same as those in the strip packing model. However, the staged pattern model utilizes height patterns to limit the total height of the used strips, unlike the strip packing model. A binary decision variable y_r for each $r \in Q_h$ represents whether or not the height pattern $\boldsymbol{b}^r$ is used. Due to constraint (13), at most one height pattern can be chosen. For all $i \in M$, constraints (12) restrict the number of used strips defined by item i to being less than or equal to the number of strips defined by the same item which is allowed by the chosen height pattern. By constraints (12), (13), and the definition of a height pattern, the total height of used strips is restricted to H.

Note that, when we use SM to solve the 2DK, we may consider only *maximal* height patterns where a height pattern $\boldsymbol{b}^r$ is called maximal if there exists no height pattern $\boldsymbol{b}^{r'}$ such that $b_k^r < b_k^{r'}$ for some $k \in M$. In SM, replacing Q_h with the set of indices of maximal height patterns can be beneficial in computational aspects since the number of variables can be reduced, while the LP relaxation bound does not change. However, for simplicity in the discussion, we consider all height patterns throughout this paper because we mainly focus on the comparison of the LP relaxation bounds of the discussed models. Also note that even if we change the inequalities in constraints (12) to equalities, the resulting model, which we call SM$^=$, is a valid formulation of the 2DK. However, SM$^=$ has two drawbacks compared to SM. Firstly, height patterns which are not maximal can not be excluded from SM$^=$ to satisfy constraints (12) at equalities. Secondly, the LP-relaxation bound of SM$^=$ is not stronger than that of SM (see the proof of Proposition 9 in Appendix A). Therefore, it is not advantageous to use SM$^=$ over SM.

The modified level packing model (ML) and the two pattern-based models (PM, SM) differ significantly in term of the definition of variables and the model size such as the number of variables and constraints. Specifically, PM and SM have exponentially many variables, whereas the number of variables of ML is pseudo-polynomial in the input size. In particular, SM has more variables than PM since the former utilizes both width and height patterns. Accordingly, these three models may yield different LP-relaxation bounds, and the computation times to obtain the bounds may vary significantly depending on the models. It is important to deal with the trade-off between the tightness of the LP-relaxation bound and the computation time in practice. Nonetheless, as mentioned in Sect. 1, this paper mainly focuses on the LP-relaxation bounds provided by these models. For the readers interested in the computational aspects of the LP-relaxations of these three models, brief computational test results are reported in Appendix B.

In the subsequent discussion, we compare the LP-relaxation bounds of the presented models: the modified level packing model (ML), the strip packing model (PM), and the staged pattern model (SM). Lodi and Monaci (2003) only computationally compared the bounds of LM and PM, while we theoretically analyze the LP-relaxation bounds of the presented three models along with brief computational results.

3 Comparison of integer programming models

For a given 2DK instance, let z^* be the optimal objective value and $z_{\mathrm{LP}}^{\mathrm{model}}$ be the LP-relaxation bound of the corresponding model for the 2DK. We also define $\mathcal{P}_{\mathrm{model}}$ as the feasible solution set of the LP-relaxation of the model. For example, for the LP-relaxation of PM, $z_{\mathrm{LP}}^{\mathrm{PM}}$ and $\mathcal{P}_{\mathrm{PM}}$ are the optimal objective value and feasible solution set, respectively.

Using these notations, we first compare the pattern-based models.

Proposition 1 $z^* \leq z_{\mathrm{LP}}^{\mathrm{SM}} \leq z_{\mathrm{LP}}^{\mathrm{PM}}$.

Proof It is clear that $z^* \leq z_{\mathrm{LP}}^{\mathrm{SM}}$ and $z^* \leq z_{\mathrm{LP}}^{\mathrm{PM}}$. Therefore, we only show that $z_{\mathrm{LP}}^{\mathrm{SM}} \leq z_{\mathrm{LP}}^{\mathrm{PM}}$. Let $(\boldsymbol{x}^{\mathrm{SM}}, \boldsymbol{y}^{\mathrm{SM}}) \in \mathcal{P}_{\mathrm{SM}}$ be an optimal solution of the LP-relaxation of SM. By constraints (12), (13), and the definition of P_h, the following inequalities

$$H \geq \max_{r \in Q_h} \sum_{i \in M} h_i b_i^r \geq \sum_{r \in Q_h} y_r^{\mathrm{SM}} \left(\sum_{i \in M} h_i b_i^r \right) \geq \sum_{i \in M} h_i \sum_{r \in Q_h} b_i^r y_r^{\mathrm{SM}} \geq \sum_{i \in M} h_i \sum_{q \in Q_w^i} x_q^{\mathrm{SM}}$$

hold. This result implies that $\boldsymbol{x}^{\mathrm{SM}} \in \mathcal{P}_{\mathrm{PM}}$ because $\boldsymbol{x}^{\mathrm{SM}}$ satisfies constraints (9) and (10) due to its definition and the above inequalities. Therefore, $z_{\mathrm{LP}}^{\mathrm{SM}}$ is less than or equal to $z_{\mathrm{LP}}^{\mathrm{PM}}$. □

We present a 2DK instance where $z_{\mathrm{LP}}^{\mathrm{SM}} < z_{\mathrm{LP}}^{\mathrm{PM}}$ in Example 3.

Now, we compare the modified level packing model and pattern-based models. For ease of the analysis, we introduce an *extended version* of PM where the width pattern set is defined for the instance $(n, H, W, \bar{\boldsymbol{h}}, \bar{\boldsymbol{w}}, \mathbb{1}, \bar{\boldsymbol{p}})$, as follows:

$$\bar{P}_w = \left\{ \bar{\boldsymbol{a}} \in \{0,1\}^n \setminus \{\boldsymbol{0}\} \ : \ \sum_{j \in N} \bar{w}_j \bar{a}_j \leq W \right\}.$$

An element of $\bar{P}_w$ is denoted as $\bar{\boldsymbol{a}}^s$ for each $s \in S_w$ where $S_w = \{1, \ldots, |\bar{P}_w|\}$. Let $\bar{\theta}(s)$ denote the minimum index in the support of $\bar{\boldsymbol{a}}^s$ for each $s \in S_w$, and $S_w^j = \{s \in S_w \ : \ \bar{\theta}(s) = j\}$ for each $j \in N$. Then, in the similar manner with Q_w, S_w can be partitioned into S_w^j's.

Example 1 Let us consider an instance $(2, 3, 3, \boldsymbol{h}, \boldsymbol{w}, \boldsymbol{d}, \boldsymbol{p})$ where $\boldsymbol{h} = (2, 1)$, $\boldsymbol{w} = (2, 1)$, $\boldsymbol{d} = (1, 2)$, and $\boldsymbol{p} = (4, 1)$. The width pattern set for this instance, P_w, is defined as follows:

$$P_w = \{(1,0), (1,1), (0,1), (0,2)\}.$$

This instance can be equivalently transformed into $(3, 3, 3, \bar{\boldsymbol{h}}, \bar{\boldsymbol{w}}, \mathbb{1}, \bar{\boldsymbol{p}})$ with $\bar{\boldsymbol{h}} = (2, 1, 1)$, $\bar{\boldsymbol{w}} = (2, 1, 1)$, and $\bar{\boldsymbol{p}} = (4, 1, 1)$. Then, $\bar{P}_w$ is defined as follows:

$$\bar{P}_w = \{(1,0,0), (1,1,0), (1,0,1), (0,1,0), (0,0,1), (0,1,1)\}.$$

Using $\bar{P}_w$, the extended version of PM, denoted as PE, is defined as follows:

$$\text{PE:} \quad \text{maximize} \quad \sum_{s \in S_w} \sum_{j \in N} \bar{p}_j \bar{a}_j^s x_s$$

$$\text{subject to} \quad \sum_{s \in S_w} \bar{a}_j^s x_s \leq 1, \quad \forall j \in N, \tag{14}$$

$$\sum_{j \in N} \sum_{s \in S_w^j} \bar{h}_j x_s \leq H, \tag{15}$$

$$x_s \in \{0, 1\}, \quad \forall s \in S_w.$$

We note that both PE and ML formulate the same instance $(n, H, W, \bar{\boldsymbol{h}}, \bar{\boldsymbol{w}}, \mathbb{1}, \bar{\boldsymbol{p}})$ which is transformed from the original one, whereas PM and SM formulate the original one. The following proposition shows the relationship between ML and PE.

Proposition 2 $z_{\text{LP}}^{\text{PE}} \leq z_{\text{LP}}^{\text{ML}}$.

Proof Let $\boldsymbol{x}^{\text{PE}} \in \mathcal{P}_{\text{PE}}$ be an optimal solution for the LP-relaxation of PE, whose objective value is $z_{\text{LP}}^{\text{PE}}$. We show that a feasible solution for the LP-relaxation of ML can be constructed from $\boldsymbol{x}^{\text{PE}}$, which yields the same objective value.

We define $\boldsymbol{x}^{\text{ML}} \in [0, 1]^{n \times n}$ as

$$x_{jk}^{\text{ML}} = \sum_{s \in S_w^k} \bar{a}_j^s x_s^{\text{PE}}, \quad \forall j \in \{k, \dots, n\}, \ \forall k \in N,$$

while $x_{jk}^{\text{ML}} = 0$, otherwise. Then, by constraints (14), we can see that, for each $j \in N$,

$$\sum_{k=1}^{j} x_{jk}^{\text{ML}} = \sum_{k=1}^{j} \sum_{s \in S_w^k} \bar{a}_j^s x_s^{\text{PE}} = \sum_{s \in S_w} \bar{a}_j^s x_s^{\text{PE}} \leq 1,$$

where the last equality holds since $\bar{a}_j^s = 0$ for each $s \in S_w^k$ such that $k \in \{j + 1, \dots, n\}$. This result implies that $\boldsymbol{x}^{\text{ML}}$ satisfies constraints (2) and (6). Furthermore, for each $k \in N$, we have

$$\begin{aligned}\sum_{j=k}^{n} \bar{w}_j x_{jk}^{\text{ML}} = \sum_{j=k}^{n} \bar{w}_j \left(\sum_{s \in S_w^k} \bar{a}_j^s x_s^{\text{PE}} \right) &= \sum_{s \in S_w^k} \left(\sum_{j=k}^{n} \bar{w}_j \bar{a}_j^s \right) x_s^{\text{PE}} \\ &\leq W \sum_{s \in S_w^k} x_s^{\text{PE}} = W x_{kk}^{\text{ML}},\end{aligned}$$

where the last inequality holds due to the definition of $\bar{P}_w$ and the last equality holds because $\bar{a}_k^s = 1$ for each $s \in S_w^k$. Therefore, $\boldsymbol{x}^{\text{ML}}$ satisfies constraints (3). Also, $\boldsymbol{x}^{\text{ML}}$ satisfies constraint (4) because

$$\sum_{k=1}^{n} \bar{h}_k x_{kk}^{\text{ML}} = \sum_{k \in N} \sum_{s \in S_w^k} \bar{h}_k x_s^{\text{PE}} \leq H,$$

by constraint (15). Finally, we have

$$x_{jk}^{\mathrm{ML}} = \sum_{s \in S_w^k} \bar{a}_j^s x_s^{\mathrm{PE}} \le \sum_{s \in S_w^k} x_s^{\mathrm{PE}} = x_{kk}^{\mathrm{ML}}$$

for each $k \in N$ and $j \in \{k, \dots, n\}$, so that even last constraints (7) hold. Therefore, $\boldsymbol{x}^{\mathrm{ML}} \in \mathcal{P}_{\mathrm{ML}}$.

On the other hand, the objective value corresponding to $\boldsymbol{x}^{\mathrm{ML}}$ is equivalent to $z_{\mathrm{LP}}^{\mathrm{PE}}$ because

$$\sum_{k=1}^{n} \sum_{j=k}^{n} \bar{p}_j x_{jk}^{\mathrm{ML}} = \sum_{k=1}^{n} \sum_{j=k}^{n} \bar{p}_j \left(\sum_{s \in S_w^k} \bar{a}_j^s x_s^{\mathrm{PE}} \right) = \sum_{k=1}^{n} \sum_{s \in S_w^k} \sum_{j=k}^{n} \bar{p}_j \bar{a}_j^s x_s^{\mathrm{PE}}$$
$$= \sum_{s \in S_w} \sum_{j \in N} \bar{p}_j \bar{a}_j^s x_s^{\mathrm{PE}},$$

where the last equality holds since

$$\bar{a}_j^s = 0, \quad \forall k \in N,\ \forall j \in \{1, \dots, k-1\},\ \forall s \in S_w^k$$

by the definition of S_w^k. Therefore, $z_{\mathrm{LP}}^{\mathrm{PE}} \le z_{\mathrm{LP}}^{\mathrm{ML}}$. □

We note that a 2DK instance where $z_{\mathrm{LP}}^{\mathrm{PE}} < z_{\mathrm{LP}}^{\mathrm{ML}}$ is presented in Example 3.

On the other hand, each element of $\bar{P}_w$ can be matched to an element of P_w through an onto function $f : \bar{P}_w \to P_w$ defined as

$$f(\bar{\boldsymbol{a}})_i = \sum_{j \in N_i} \bar{a}_j, \quad \forall i \in M,$$

where $N_i = \{j \in N : \beta_j = i\}$ for each $i \in M$. We note that $|N_i| = d_i$.For instance, in Example 1, $f(1,1,0) = f(1,0,1) = (1,1)$. Let $S_w(q) = \{s \in S_w : f(\bar{\boldsymbol{a}}^s) = \boldsymbol{a}^q\}$ for each $q \in Q_w$. Then, S_w can also be partitioned into $S_w(q)$'s, that is, $S_w = \cup_{q \in Q_w} S_w(q)$ and $S_w(q_1) \cap S_w(q_2) = \emptyset$ for any $q_1, q_2 \in Q_w$ such that $q_1 \ne q_2$. Additionally, it can be easily shown that

$$|S_w(q)| = \prod_{i \in M} C(d_i, a_i^q), \quad \forall q \in Q_w, \tag{16}$$

from the definition of $S_w(q)$'s, where $C(\gamma_1, \gamma_2) = \gamma_1!/(\gamma_2!(\gamma_1 - \gamma_2)!)$ for some $\gamma_1 \in \mathbb{Z}_+$ and $\gamma_2 \in \mathbb{Z}_+$ such that $\gamma_1 \ge \gamma_2$. It means the number of combinations of γ_1 items taken γ_2 at a time.

Based on these observations, we show that PE provides a bridge to compare the LP-relaxation bounds between ML and PM.

Proposition 3 $z_{\mathrm{LP}}^{\mathrm{PM}} = z_{\mathrm{LP}}^{\mathrm{PE}}$.

Proof Let $\boldsymbol{x}^{\mathrm{PM}} \in \mathcal{P}_{\mathrm{PM}}$ be an optimal solution of the LP-relaxation of PM, which yields the objective value $z_{\mathrm{LP}}^{\mathrm{PM}}$. We first show that $z_{\mathrm{LP}}^{\mathrm{PE}} \ge z_{\mathrm{LP}}^{\mathrm{PM}}$ by constructing a feasible solution of the LP-relaxation of PE from $\boldsymbol{x}^{\text{PM}}$, which yields the same objective value.

Let us define $\boldsymbol{x}^{\text{PE}} \in [0,1]^{|S_w|}$ as $x_s^{\text{PE}} = x_q^{\text{PM}}/|S_w(q)|$ for each $q \in Q_w$ and $s \in S_w(q)$. For any fixed $q \in Q_w$ and i belonging to the support of $\boldsymbol{a}^q$, the following inequality holds by constraints (9):

$$x_q^{\text{PM}} \le \frac{d_i}{a_i^q}. \tag{17}$$

Furthermore, for such q and i, we have the following inequalities from equality (16):

$$|S_w(q)| \ge C(d_i, a_i^q) = \frac{d_i \times (d_i - 1) \times \cdots \times (d_i - a_i^q + 1)}{a_i^q \times (a_i^q - 1) \times \cdots \times 1} \ge \frac{d_i}{a_i^q}, \tag{18}$$

where the last inequality holds since $d_i \ge a_i^q$. Inequalities (17) and (18) imply that $x_s^{\text{PE}} \in [0,1]$ for each $q \in Q_w$ and $s \in S_w(q)$ because

$$x_s^{\text{PE}} = \frac{x_q^{\text{PM}}}{|S_w(q)|} \le \frac{d_i}{a_i^q} \cdot \frac{a_i^q}{d_i} = 1.$$

Now, we show that $\boldsymbol{x}^{\text{PE}}$ satisfies constraints (14) and (15). Let us consider constraint (14) corresponding to some $j \in N$, and let $j \in N_i$ for some $i \in M$. The left hand-side of this constraint for $\boldsymbol{x}^{\text{PE}}$ can be represented as follows:

$$\sum_{s \in S_w} \bar{a}_j^s x_s^{\text{PE}} = \sum_{q \in Q_w} \sum_{s \in S_w(q)} \bar{a}_j^s x_s^{\text{PE}} = \sum_{q \in Q_w} \frac{x_q^{\text{PM}}}{|S_w(q)|} \sum_{s \in S_w(q)} \bar{a}_j^s.$$

Here, for any fixed $q \in Q_w$, $\sum_{s \in S_w(q)} \bar{a}_j^s$ is equivalent to the number of elements s in $S_w(q)$ where $\bar{a}_j^s = 1$. We can see that the number of such elements is equal to the size of $S_w(q')$ in a modified instance with a reduced demand of the item type i to $d_i - 1$ for some $i \in M$ such that $j \in N_i$, where q' is the index of the width pattern such that $a_i^{q'} = a_i^q - 1$ and $a_k^{q'} = a_k^q$ for each $k \in M \setminus \{i\}$. Therefore, $\sum_{s \in S_w(q)} \bar{a}_j^s$ can be computed as follows:

$$\begin{aligned}\sum_{s \in S_w(q)} \bar{a}_j^s &= |\{s \in S_w(q) : \bar{a}_j^s = 1\}| \\ &= C(d_i - 1, a_i^q - 1) \prod_{l \in M \setminus \{i\}} C(d_l, a_l^q) = \frac{a_i^q}{d_i} |S_w(q)|.\end{aligned}$$

This result implies that $\boldsymbol{x}^{\text{PE}}$ satisfies constraints (14) because, for each $i \in M$ and $j \in N_i$,

$$\sum_{s \in S_w} \bar{a}_j^s x_s^{\text{PE}} = \sum_{q \in Q_w} \frac{x_q^{\text{PM}}}{|S_w(q)|} \sum_{s \in S_w(q)} \bar{a}_j^s = \sum_{q \in Q_w} \frac{a_i^q}{d_i} x_q^{\text{PM}} \le 1,$$

where the last inequality holds due to constraints (9). Let us consider now constraint (15). From the definition of S_w^j's and $S_w(q)$'s, it can be easily shown that

$$\bigcup_{q \in Q_w^i} S_w(q) = \bigcup_{j \in N_i} S_w^j, \quad \forall i \in M.$$

By utilizing this relationship, we can see that

$$\begin{aligned} \sum_{j \in N} \sum_{s \in S_w^j} \bar{h}_j x_s^{\mathrm{PE}} &= \sum_{i \in M} \sum_{j \in N_i} \sum_{s \in S_w^j} \bar{h}_j x_s^{\mathrm{PE}} = \sum_{i \in M} \sum_{q \in Q_w^i} h_i \left(\sum_{s \in S_w(q)} x_s^{\mathrm{PE}} \right) \\ &= \sum_{i \in M} \sum_{q \in Q_w^i} h_i x_q^{\mathrm{PM}} \leq H, \end{aligned} \tag{19}$$

where the last inequality holds due to constraint (10). On the other hand, $\boldsymbol{x}^{\mathrm{PE}}$ yields the same objective value as $z_{\mathrm{LP}}^{\mathrm{PM}}$. By the definition of $S_w(q)$'s, the following equalities hold:

$$\sum_{i \in M} p_i a_i^q = \sum_{j \in N} \bar{p}_j \bar{a}_j^s, \quad \forall q \in Q_w,\ \forall s \in S_w(q).$$

Then, the objective value corresponding to $\boldsymbol{x}^{\mathrm{PE}}$ can be represented as

$$\sum_{q \in Q_w} \sum_{s \in S_w(q)} \sum_{j \in N} \bar{p}_j \bar{a}_j^s x_s^{\mathrm{PE}} = \sum_{q \in Q_w} \sum_{s \in S_w(q)} \sum_{i \in M} p_i a_i^q x_s^{\mathrm{PE}} = \sum_{q \in Q_w} \sum_{i \in M} p_i a_i^q x_q^{\mathrm{PM}}, \tag{20}$$

where the last equality implicitly uses a step from x_s^{PE} to x_q^{PM} as the last step in (19). The last term of (20) is equivalent to $z_{\mathrm{LP}}^{\mathrm{PM}}$. This result implies that $z_{\mathrm{LP}}^{\mathrm{PM}} \leq z_{\mathrm{LP}}^{\mathrm{PE}}$.

Finally, we show that $z_{\mathrm{LP}}^{\mathrm{PE}} \leq z_{\mathrm{LP}}^{\mathrm{PM}}$. Let $\boldsymbol{x}^{\mathrm{PE}}$ be a given optimal solution of the LP-relaxation of PE, which yields the objective value $z_{\mathrm{LP}}^{\mathrm{PE}}$. We define $\boldsymbol{x}^{\mathrm{PM}} \in \mathbb{R}_+^{|Q_w|}$ as $x_q^{\mathrm{PM}} = \sum_{s \in S_w(q)} x_s^{\mathrm{PE}}$ for each $q \in Q_w$. Then, $\boldsymbol{x}^{\mathrm{PM}}$ satisfies constraints (9) because, for any $i \in M$,

$$\begin{aligned} \sum_{q \in Q_w} a_i^q x_q^{\mathrm{PM}} &= \sum_{q \in Q_w} \sum_{s \in S_w(q)} a_i^q x_s^{\mathrm{PE}} = \sum_{q \in Q_w} \sum_{s \in S_w(q)} \sum_{j \in N_i} \bar{a}_j^s x_s^{\mathrm{PE}} \\ &= \sum_{j \in N_i} \sum_{s \in S_w} \bar{a}_j^s x_s^{\mathrm{PE}} \leq d_i, \end{aligned}$$

where the last inequality is derived by the summation of constraints (14) corresponding to $j \in N_i$ and by the size of N_i itself. Additionally, $\boldsymbol{x}^{\mathrm{PM}}$ satisfies constraint (10) due to the equalities in (19) to be followed in reverse order, and constraint (15) that assures the validity of H as an upper bound. Also, $\boldsymbol{x}^{\mathrm{PM}}$ yields the same objective value with $\boldsymbol{x}^{\mathrm{PE}}$ because of the equalities of (20) that can be developed in reverse order. These results imply that $\boldsymbol{x}^{\mathrm{PM}} \in \mathcal{P}_{\mathrm{PM}}$ and $z_{\mathrm{LP}}^{\mathrm{PE}} \leq z_{\mathrm{LP}}^{\mathrm{PM}}$. Therefore, the result follows. □

With Propositions 1 and 2, Proposition 3 states that the pattern-based models, PM and SM, provide tighter LP-relaxation bounds compared to ML and LM. In other words, $z_{\text{LP}}^{\text{PM}}$ and $z_{\text{LP}}^{\text{SM}}$ are lower bounds on $z_{\text{LP}}^{\text{ML}}$ and $z_{\text{LP}}^{\text{LM}}$. In the subsequent discussion, we also provide upper bounds on $z_{\text{LP}}^{\text{ML}}$ using $z_{\text{LP}}^{\text{PM}}$ and $z_{\text{LP}}^{\text{SM}}$.

Let us introduce sub-structures of $\mathcal{P}_{\text{ML}}$, which have useful properties to derive upper bounds on $z_{\text{LP}}^{\text{ML}}$. For each $k \in N$, let us define a polytope $\mathcal{R}_k \subseteq \mathbb{R}_+^n$, where its generic element is denoted as $\boldsymbol{x}_k = (x_{1k}, \ldots, x_{nk})$, as follows:

$$\mathcal{R}_k = \left\{ \boldsymbol{x}_k \in [0,1]^n \ : \ \begin{array}{l} \sum_{j=k+1}^{n} \bar{w}_j x_{jk} \le (W - \bar{w}_k) x_{kk}, \\ x_{jk} \le x_{kk}, \quad \forall j \in \{k+1, \ldots, n\}, \\ x_{jk} = 0, \quad \forall j \in \{1, \ldots, k-1\} \end{array} \right\}.$$

We note that constraints in $\mathcal{R}_k$ correspond to constraints (3), (5), and (7) in ML for each $k \in N$. For any $\boldsymbol{x}^{\text{ML}} \in \mathcal{P}_{\text{ML}} \subseteq \mathbb{R}_+^{n \times n}$, let $\boldsymbol{x}_k^{\text{ML}} = (x_{1k}^{\text{ML}}, \ldots, x_{nk}^{\text{ML}})$ for each $k \in N$. By definition, it is clear that $\boldsymbol{x}_k^{\text{ML}} \in \mathcal{R}_k$ for any fixed $k \in N$. Therefore, $\boldsymbol{x}_k^{\text{ML}}$ can be represented as a convex combination of the extreme points of $\mathcal{R}_k$. Let $\boldsymbol{v}_k^i = (v_{1k}^i, \ldots, v_{nk}^i)$ for all $i \in V_k$ be the all non-zero extreme points of $\mathcal{R}_k$, where V_k denotes the index set of them. Of course, $v_{jk}^i = 0$ for each $j \in \{1, \ldots, k-1\}$. Then, $\boldsymbol{x}_k^{\text{ML}}$ can be represented as $\boldsymbol{x}_k^{\text{ML}} = \sum_{i \in V_k} \lambda^i \boldsymbol{v}_k^i$ for some $\lambda^i \in [0,1]$ for each $i \in V_k$ such that $\sum_{i \in V_k} \lambda^i \le 1$, since the zero extreme point is not considered.

Proposition 4 *For each $k \in N$ and $i \in V_k$, $\boldsymbol{v}_k^i$ has at most one fractional component, while $v_{kk}^i = 1$ and $v_{jk}^i = 0$ for each $j \in \{1, \ldots, k-1\}$.*

Proof For any fixed $k \in N$, let $\hat{\boldsymbol{x}}_k \in \mathcal{R}_k$ be a non-zero extreme point of $\mathcal{R}_k$. Since $\hat{\boldsymbol{x}}_k \in \mathcal{R}_k$, $\hat{x}_{jk} = 0$ for each $j \in \{1, \ldots, k-1\}$. We only consider the case when $\hat{x}_{kk} > 0$ since $\hat{\boldsymbol{x}}_k = \boldsymbol{0}$ if $\hat{x}_{kk} = 0$ by the definition of $\mathcal{R}_k$. Suppose that $0 < \hat{x}_{kk} < 1$. It can be easily shown that $\hat{\boldsymbol{x}}_k / \hat{x}_{kk} \in \mathcal{R}_k$. This result implies that $\hat{\boldsymbol{x}}_k$ with $0 < \hat{x}_{kk} < 1$ cannot be an extreme point of $\mathcal{R}_k$ because such $\hat{\boldsymbol{x}}_k$ is represented as a convex combination of $\boldsymbol{0}$ and $\hat{\boldsymbol{x}}_k / \hat{x}_{kk}$. Finally, assume that $\hat{x}_{kk} = 1$, and let $\mathcal{R}_k'$ be a facet of $\mathcal{R}_k$, defined as $\mathcal{R}_k \cap \{\boldsymbol{x}_k \in [0,1]^n \ : \ x_{kk} = 1\}$. It is clear that $\hat{\boldsymbol{x}}_k$ is an extreme point of $\mathcal{R}_k$ if and only if $\hat{\boldsymbol{x}}_k$ is an extreme point of $\mathcal{R}_k'$ by the definition of a facet. On the other hand, $\mathcal{R}_k'$ can be expressed as follows:

$$\mathcal{R}_k' = \left\{ \boldsymbol{x}_k \in [0,1]^n \ : \ \begin{array}{l} \sum_{j=k+1}^{n} \bar{w}_j x_{jk} \le W - \bar{w}_k, \\ x_{kk} = 1, \\ x_{jk} = 0, \quad \forall j \in \{1, \ldots, k-1\} \end{array} \right\}.$$

Here, constraints $x_{jk} \le x_{kk}$ for $j \in \{k+1, \ldots, n\}$ are dropped since they are redundant to the bound constraints $x_{jk} \le 1$ for $j \in \{k+1, \ldots, n\}$. We can see that $\mathcal{R}_k'$ is represented as the LP-relaxation of the feasible solution set of a binary knapsack problem, where an extreme point has at most one fractional component. Accordingly, an extreme point of $\mathcal{R}_k'$ has at most one fractional component among x_{jk}'s where $j \in \{k+1, \dots, n\}$, and $\hat{\boldsymbol{x}}_k$ does so. Therefore, the result follows. □

Let $\lfloor \boldsymbol{v}_k^i \rfloor = (\lfloor v_{1k}^i \rfloor, \dots, \lfloor v_{nk}^i \rfloor)$ and $\lceil \boldsymbol{v}_k^i \rceil = (\lceil v_{1k}^i \rceil, \dots, \lceil v_{nk}^i \rceil)$ for each $i \in V_k$ and $k \in N$. Then, $\lfloor \boldsymbol{v}_k^i \rfloor \neq \boldsymbol{0}$ and $\lfloor \boldsymbol{v}_k^i \rfloor$ corresponds to a width pattern $\bar{\boldsymbol{a}}^s$ of $\bar{P}_w$ for some $s \in S_w^k$ by Proposition 4 and the definition of $\mathcal{R}_k$. On the other hand, let $u(k) \in S_w$ be the index of a width pattern in $\bar{P}_w$, represented as a unit vector with $\bar{a}_k^{u(k)} = 1$ for each $k \in N$. It is clear that $u(k) \in S_w^k$ for each $k \in N$. Then, for any fixed $k \in N$ and $i \in V_k$, $\lceil \boldsymbol{v}_k^i \rceil - \lfloor \boldsymbol{v}_k^i \rfloor$ represents a zero vector or a unit vector corresponding to a width pattern $\bar{\boldsymbol{a}}^{u(l)}$ of $\bar{P}_w$ for some $l \in \{k+1, \dots, n\}$ since $v_{kk}^i = 1$ and $v_{jk}^i = 0$ for each $j \in \{1, \dots, k-1\}$ by Proposition 4. Therefore, $u(l) \notin S_w^k$ for each $l > k$. From this relationship between $\boldsymbol{v}_k^i$'s and width patterns, we present the upper bound on $z_{\mathrm{LP}}^{\mathrm{ML}}$ using $z_{\mathrm{LP}}^{\mathrm{PE}}$ in the following proposition.

Proposition 5 $z_{\mathrm{LP}}^{\mathrm{ML}} \leq 2 z_{\mathrm{LP}}^{\mathrm{PE}}$.

Proof Let $\boldsymbol{x}^{\mathrm{ML}} \in \mathcal{P}_{\mathrm{ML}}$ be an optimal solution for the LP-relaxation of ML, where the corresponding objective value is $z_{\mathrm{LP}}^{\mathrm{ML}}$. We show that a feasible solution for the LP-relaxation of PE can be constructed from $\boldsymbol{x}^{\mathrm{ML}}$, where the corresponding objective value is greater than or equal to $(1/2) z_{\mathrm{LP}}^{\mathrm{ML}}$.

Recall that, for any fixed $k \in N$, $\boldsymbol{x}_k^{\mathrm{ML}} = (x_{1k}^{\mathrm{ML}}, \dots, x_{nk}^{\mathrm{ML}})$ can be represented with the extreme points of $\mathcal{R}_k$. Let $\boldsymbol{x}_k^{\mathrm{ML}} = \sum_{i \in V_k} \lambda_k^i \boldsymbol{v}_k^i$ for some $\lambda_k^i \in [0,1]$ for each $i \in V_k$ such that $\sum_{i \in V_k} \lambda_k^i \leq 1$. We note that $\boldsymbol{v}_k^i$ for each $i \in V_k$ can be rewritten as

$$\boldsymbol{v}_k^i = \lfloor \boldsymbol{v}_k^i \rfloor + \boldsymbol{v}_k^i - \lfloor \boldsymbol{v}_k^i \rfloor = \lfloor \boldsymbol{v}_k^i \rfloor + f_k^i (\lceil \boldsymbol{v}_k^i \rceil - \lfloor \boldsymbol{v}_k^i \rfloor) \tag{21}$$

where $f_k^i = \sum_{j \in N} (v_{jk}^i - \lfloor v_{jk}^i \rfloor)$, because $\boldsymbol{v}_k^i - \lfloor \boldsymbol{v}_k^i \rfloor$ can have at most one non-zero component by Proposition 4.

For each $k \in N$, V_k can be partitioned into V_k^D and V_k^U where

$$V_k^D = \left\{ i \in V_k \, : \, \sum_{j \in N} \bar{p}_j \lfloor v_{jk}^i \rfloor \geq \sum_{j \in N} \bar{p}_j f_k^i (\lceil v_{jk}^i \rceil - \lfloor v_{jk}^i \rfloor) \right\},$$

and

$$V_k^U = \left\{ i \in V_k \, : \, \sum_{j \in N} \bar{p}_j \lfloor v_{jk}^i \rfloor < \sum_{j \in N} \bar{p}_j f_k^i (\lceil v_{jk}^i \rceil - \lfloor v_{jk}^i \rfloor) \right\}.$$

By definition, V_k^D contains at least all the indexes for which $\boldsymbol{v}_k^i$ is an integral vector, therefore V_k^U is empty if all the extreme points are integer. We note that V_k^U can be also empty even if some extreme points contain a fractional component.

For any two real-valued n-dimensional vectors $\boldsymbol{\omega}^1, \boldsymbol{\omega}^2$ where $\boldsymbol{\omega}^i = (\omega_1^i, \dots, \omega_n^i)$ for $i = 1, 2$, let us denote $\boldsymbol{\omega}^1 \geq \boldsymbol{\omega}^2$ if $\omega_j^1 \geq \omega_j^2$ for all $j = 1, \dots, n$. Using this notation, we have the following inequalities for each $k \in N$:

$$\boldsymbol{x}_k^{\text{ML}} = \sum_{i \in V_k^D} \lambda_k^i \boldsymbol{v}_k^i + \sum_{i \in V_k^U} \lambda_k^i \boldsymbol{v}_k^i \geq \sum_{i \in V_k^D} \lambda_k^i \lfloor \boldsymbol{v}_k^i \rfloor + \sum_{i \in V_k^U} \lambda_k^i f_k^i (\lceil \boldsymbol{v}_k^i \rceil - \lfloor \boldsymbol{v}_k^i \rfloor), \tag{22}$$

where the last inequality holds by the equality (21).

As mentioned earlier, for each $k \in N$, $\lfloor \boldsymbol{v}_k^i \rfloor$ for each $i \in V_k$ corresponds to a width pattern $\bar{\boldsymbol{a}}^s$ of $\bar{P}_w$ for some $s \in S_w^k$. In addition, $\lceil \boldsymbol{v}_k^i \rceil - \lfloor \boldsymbol{v}_k^i \rfloor$ for each $i \in V_k^U$ corresponds to $\bar{\boldsymbol{a}}^{u(l)}$ for some $l \in \{k+1, \dots, n\}$ because $\lceil \boldsymbol{v}_k^i \rceil - \lfloor \boldsymbol{v}_k^i \rfloor \neq \mathbf{0}$ by the definition of V_k^U. Based on these observations, we further partition V_k^D and V_k^U.

Let us define $D_k(s) = \{i \in V_k^D : \bar{\boldsymbol{a}}^s = \lfloor \boldsymbol{v}_k^i \rfloor\}$ for each $s \in S_w^k$, and let $U_k(l) = \{i \in V_k^U : \bar{\boldsymbol{a}}^{u(l)} = \lceil \boldsymbol{v}_k^i \rceil - \lfloor \boldsymbol{v}_k^i \rfloor\}$ for each $l \in \{k+1, \dots, n\}$. It is clear that V_k^D and V_k^U are partitioned into $D_k(s)$'s and $U_k(l)$'s, respectively. We note that $D_k(s)$'s and $U_k(l)$'s may contain more than one element. For example, when $n = 5$ and $k = 1$, suppose that a width pattern $\bar{\boldsymbol{a}}^s = (1, 1, 0, 0, 0)$ is given for some $s \in S_w^1$ with four vectors, $\boldsymbol{v}_1^{i_1} = (1, 1, 0.5, 0, 0)$, $\boldsymbol{v}_1^{i_2} = (1, 1, 0, 0.5, 0)$, $\boldsymbol{v}_1^{i_3} = (1, 0, 1, 0.5, 0)$, and $\boldsymbol{v}_1^{i_4} = (1, 0, 0, 0.5, 1)$, where $i_1, i_2 \in V_1^D$ and $i_3, i_4 \in V_1^U$. Then, $i_1, i_2 \in D_1(s)$ for such s since $\bar{\boldsymbol{a}}^s = (1, 1, 0, 0, 0) = \lfloor \boldsymbol{v}_1^{i_1} \rfloor = \lfloor \boldsymbol{v}_1^{i_2} \rfloor$. Additionally, $\bar{\boldsymbol{a}}^{u(4)} = (0, 0, 0, 1, 0) = \lceil \boldsymbol{v}_1^{i_3} \rceil - \lfloor \boldsymbol{v}_1^{i_3} \rfloor = \lceil \boldsymbol{v}_1^{i_4} \rceil - \lfloor \boldsymbol{v}_1^{i_4} \rfloor$, that is, $i_3, i_4 \in U_1(4)$.

We then define $\mu_k^s = \sum_{i \in D_k(s)} \lambda_k^i$ for each $s \in S_w^k$ and $\pi_k^l = \sum_{i \in U_k(l)} \lambda_k^i f_k^i$ for each $l \in \{k+1, \dots, n\}$. If $D_k(s) = \emptyset$ for some $s \in S_w^k$ or $U_k(l) = \emptyset$ for some $l \in \{k+1, \dots, n\}$, we set μ_k^s or π_k^l to 0. Then, we can see that

$$\begin{aligned} 1 \geq x_{kk}^{\text{ML}} &= \sum_{i \in V_k} \lambda_k^i v_{kk}^i = \sum_{i \in V_k} \lambda_k^i = \sum_{i \in V_k^D} \lambda_k^i + \sum_{i \in V_k^U} \lambda_k^i \\ &\geq \sum_{s \in S_w^k} \sum_{i \in D_k(s)} \lambda_k^i + \sum_{l=k+1}^{n} \sum_{i \in U_k(l)} \lambda_k^i f_k^i = \sum_{s \in S_w^k} \mu_k^s + \sum_{l=k+1}^{n} \pi_k^l, \end{aligned} \tag{23}$$

where the second equality holds since $v_{kk}^i = 1$ for each $i \in V_k$ by Proposition 4 and the second inequality holds since $f_k^i < 1$ for each $i \in V_k$. In addition, the following result can be derived from the inequality (22):

$$\begin{aligned} \boldsymbol{x}_k^{\text{ML}} &\geq \sum_{i \in V_k^D} \lambda_k^i \lfloor \boldsymbol{v}_k^i \rfloor + \sum_{i \in V_k^U} \lambda_k^i f_k^i (\lceil \boldsymbol{v}_k^i \rceil - \lfloor \boldsymbol{v}_k^i \rfloor) \\ &= \sum_{s \in S_w^k} \sum_{i \in D_k(s)} \lambda_k^i \bar{\boldsymbol{a}}^s + \sum_{l=k+1}^{n} \sum_{i \in U_k(l)} \lambda_k^i f_k^i \bar{\boldsymbol{a}}^{u(l)} \\ &= \sum_{s \in S_w^k} \mu_k^s \bar{\boldsymbol{a}}^s + \sum_{l=k+1}^{n} \pi_k^l \bar{\boldsymbol{a}}^{u(l)} \end{aligned}$$

where the first equality holds since $\lfloor \boldsymbol{v}_k^i \rfloor = \bar{\boldsymbol{a}}^s$ for each $s \in S_w^k$ and $i \in D_k(s)$, while $\lceil \boldsymbol{v}_k^i \rceil - \lfloor \boldsymbol{v}_k^i \rfloor = \bar{\boldsymbol{a}}^{u(l)}$ for each $l \in \{k+1, \ldots, n\}$ and $i \in U_k(l)$. Then, we can see that

$$x_{jk}^{\mathrm{ML}} \geq \sum_{s \in S_w^k} \mu_k^s \bar{a}_j^s + \sum_{l=k+1}^{n} \pi_k^l \bar{a}_j^{u(l)} = \sum_{s \in S_w^k \setminus \{u(k)\}} \mu_k^s \bar{a}_j^s + \pi_k^j \tag{24}$$

for each $j \in \{k+1, \ldots, n\}$ because $\bar{a}_j^{u(l)} = 1$ if and only if $l = j$.

Now, let us define $\boldsymbol{x}^{\mathrm{PE}} \in \mathbb{R}_+^{|S_w|}$ as

$$x_s^{\mathrm{PE}} = \begin{cases} \mu_k^s, & \text{if } s \in S_w^k \setminus \{u(k)\} \\ \mu_k^{u(k)} + \sum_{\tau=1}^{k-1} \pi_\tau^k, & \text{if } s = u(k) \end{cases}$$

for each $k \in N$ and $s \in S_w^k$. Here, the second term in case $s = u(k)$ is a sum of π_τ^k's for $\tau \in \{1, \ldots, k-1\}$, where π_τ^k is not zero if and only if $\bar{\boldsymbol{a}}^{u(k)}$ corresponds to $\lceil \boldsymbol{v}_\tau^i \rceil - \lfloor \boldsymbol{v}_\tau^i \rfloor$ for some $i \in V_\tau^U$. We show that $\boldsymbol{x}^{\mathrm{PE}} \in \mathcal{P}_{\mathrm{PE}}$. Firstly, the left hand-side of constraints (14) in PE for some $j \in N$ can be rewritten with $\boldsymbol{x}^{\mathrm{PE}}$ as follows:

$$\begin{aligned}
\sum_{s \in S_w} \bar{a}_j^s x_s^{\mathrm{PE}} &= \sum_{k=1}^{j} \sum_{s \in S_w^k} \bar{a}_j^s x_s^{\mathrm{PE}} = \sum_{k=1}^{j} \left(\sum_{s \in S_w^k \setminus \{u(k)\}} \bar{a}_j^s \mu_k^s + \bar{a}_j^{u(k)} x_{u(k)}^{\mathrm{PE}} \right) \\
&= \sum_{k=1}^{j} \sum_{s \in S_w^k \setminus \{u(k)\}} \bar{a}_j^s \mu_k^s + \bar{a}_j^{u(j)} x_{u(j)}^{\mathrm{PE}} \\
&= \sum_{k=1}^{j} \sum_{s \in S_w^k \setminus \{u(k)\}} \bar{a}_j^s \mu_k^s + \mu_j^{u(j)} + \sum_{\tau=1}^{j-1} \pi_\tau^j \\
&= \sum_{k=1}^{j-1} \left(\sum_{s \in S_w^k \setminus \{u(k)\}} \bar{a}_j^s \mu_k^s + \pi_k^j \right) + \sum_{s \in S_w^j} \bar{a}_j^s \mu_j^s.
\end{aligned}$$

Here, the first equality holds since $\bar{a}_j^s = 0$ for each $k \in \{j+1, \ldots, n\}$ and $s \in S_w^k$, while the third equality holds since $\bar{a}_j^{u(k)} = 1$ if and only if $k = j$, where $\bar{a}_j^{u(j)} = 1$ also supports the last equality. Then, from inequalities (23) and (24), we can obtain the following result:

$$\sum_{s \in S_w} \bar{a}_j^s x_s^{\mathrm{PE}} = \sum_{k=1}^{j-1} \left(\sum_{s \in S_w^k \setminus \{u(k)\}} \bar{a}_j^s \mu_k^s + \pi_k^j \right) + \sum_{s \in S_w^j} \bar{a}_j^s \mu_j^s \leq \sum_{k=1}^{j-1} x_{jk}^{\mathrm{ML}} + x_{jj}^{\mathrm{ML}} \leq 1,$$

where the last inequality holds due to constraint (2). Here, we note that $x_{jj}^{\mathrm{ML}} \geq \sum_{s \in S_w^j} \bar{a}_j^s \mu_j^s$ holds by the inequality (23) since $\bar{a}_j^s = 1$ when $s \in S_w^j$. Therefore, $\boldsymbol{x}^{\mathrm{PE}}$ satisfies constraints (14) in PE. In addition, this result implies that $x_s^{\mathrm{PE}} \leq 1$ for each $s \in S_w$. Next, from the left hand-side of constraint (15) with $\boldsymbol{x}^{\mathrm{PE}}$, we can derive the following inequalities:

$$\begin{aligned}\sum_{k=1}^{n}\sum_{s\in S_w^k}\bar{h}_k x_s^{\mathrm{PE}} &= \sum_{k=1}^{n}\left(\sum_{s\in S_w^k}\bar{h}_k\mu_k^s + \sum_{\tau=1}^{k-1}\bar{h}_k\pi_\tau^k\right)\\ &\leq \sum_{k=1}^{n}\left(\sum_{s\in S_w^k}\bar{h}_k\mu_k^s + \sum_{\tau=1}^{k-1}\bar{h}_\tau\pi_\tau^k\right) = \sum_{k=1}^{n}\bar{h}_k\left(\sum_{s\in S_w^k}\mu_k^s + \sum_{l=k+1}^{n}\pi_k^l\right),\end{aligned}$$

where the inequality holds since $\bar{h}_k \leq \bar{h}_l$ for each $l \in \{1,\ldots,k-1\}$. Then, $\boldsymbol{x}^{\mathrm{PE}}$ satisfies constraint (15) by the inequality (23), because

$$\sum_{k=1}^{n}\sum_{s\in S_w^k}\bar{h}_k x_s^{\mathrm{PE}} \leq \sum_{k=1}^{n}\bar{h}_k\left(\sum_{s\in S_w^k}\mu_k^s + \sum_{l=k+1}^{n}\pi_k^l\right) \leq \sum_{k=1}^{n}\bar{h}_k x_{kk}^{\mathrm{ML}} \leq H,$$

where the last inequality holds due to constraint (4). Therefore, $\boldsymbol{x}^{\mathrm{PE}} \in \mathcal{P}_{\mathrm{PE}}$.

We now consider the objective value corresponding to $\boldsymbol{x}^{\mathrm{PE}}$, denoted as $\hat{z}$. Let $\bar{p}^s = \sum_{j\in N}\bar{p}_j\bar{a}_j^s$ for each $s \in S_w$. Then, $\hat{z}$ can be represented as follows:

$$\begin{aligned}\hat{z} = \sum_{k=1}^{n}\sum_{s\in S_w^k}\bar{p}^s x_s^{\mathrm{PE}} &= \sum_{k=1}^{n}\left(\sum_{s\in S_w^k\setminus\{u(k)\}}\bar{p}^s\mu_k^s + \bar{p}^{u(k)}\left(\mu_k^{u(k)} + \sum_{\tau=1}^{k-1}\pi_\tau^k\right)\right)\\ &= \sum_{k=1}^{n}\left(\sum_{s\in S_w^k}\bar{p}^s\mu_k^s + \bar{p}^{u(k)}\sum_{\tau=1}^{k-1}\pi_\tau^k\right) = \sum_{k=1}^{n}\left(\sum_{s\in S_w^k}\bar{p}^s\mu_k^s + \sum_{l=k+1}^{n}\bar{p}^{u(l)}\pi_k^l\right)\\ &= \sum_{k=1}^{n}\left(\sum_{s\in S_w^k}\sum_{i\in D_k(s)}\bar{p}^s\lambda_k^i + \sum_{l=k+1}^{n}\sum_{i\in U_k(l)}\bar{p}^{u(l)}\lambda_k^i f_k^i\right).\end{aligned}$$

Here, $\bar{p}^s = \sum_{j\in N}\bar{p}_j\lfloor v_{jk}^i\rfloor$ for each $k \in N$, $s \in S_w^k$, and $i \in D_k(s)$ by the definition of $D_k(s)$. In addition, $\bar{p}^{u(l)} = \sum_{j\in N}\bar{p}_j(\lceil v_{jk}^i\rceil - \lfloor v_{jk}^i\rfloor)$ for each $k \in N$, $l \in \{k+1,\ldots,n\}$, and $i \in U_k(l)$ by the definition of $U_k(l)$. Accordingly, following equalities hold:

$$\begin{aligned}\hat{z} &= \sum_{k=1}^{n}\left(\sum_{s\in S_w^k}\sum_{i\in D_k(s)}\sum_{j\in N}\bar{p}_j\lfloor v_{jk}^i\rfloor\lambda_k^i + \sum_{l=k+1}^{n}\sum_{i\in U_k(l)}\sum_{j\in N}\bar{p}_j(\lceil v_{jk}^i\rceil - \lfloor v_{jk}^i\rfloor)\lambda_k^i f_k^i\right)\\ &= \sum_{k=1}^{n}\left(\sum_{i\in V_k^D}\sum_{j\in N}\bar{p}_j\lfloor v_{jk}^i\rfloor\lambda_k^i + \sum_{i\in V_k^U}\sum_{j\in N}\bar{p}_j(\lceil v_{jk}^i\rceil - \lfloor v_{jk}^i\rfloor)\lambda_k^i f_k^i\right)\\ &= \sum_{k=1}^{n}\sum_{i\in V_k}\lambda_k^i \max\left\{\sum_{j\in N}\bar{p}_j\lfloor v_{jk}^i\rfloor, \sum_{j\in N}\bar{p}_j f_k^i(\lceil v_{jk}^i\rceil - \lfloor v_{jk}^i\rfloor)\right\},\end{aligned}$$

where the last equality holds by the definition of V_k^D and V_k^U for each $k \in N$. On the other hand, $z_{\text{LP}}^{\text{ML}}$ can be expressed as

$$z_{\text{LP}}^{\text{ML}} = \sum_{k=1}^{n} \sum_{j=k}^{n} \bar{p}_j x_{jk}^{\text{ML}} = \sum_{k=1}^{n} \sum_{j \in N} \sum_{i \in V_k} \bar{p}_j \lambda_k^i v_{jk}^i$$
$$= \sum_{k=1}^{n} \sum_{i \in V_k} \lambda_k^i \sum_{j \in N} \bar{p}_j \Big(\lfloor v_{jk}^i \rfloor + f_k^i(\lceil v_{jk}^i \rceil - \lfloor v_{jk}^i \rfloor) \Big),$$

where the last equality comes from (21). Therefore, it is clear that $z_{\text{LP}}^{\text{ML}} \le 2\hat{z}$. This result implies that $z_{\text{LP}}^{\text{ML}} \le 2z_{\text{LP}}^{\text{PE}}$ since $\hat{z} \le z_{\text{LP}}^{\text{PE}}$. □

Propositions 3 and 5 imply that, even though ML provides the weaker LP-relaxation bound compared to PM, the bound is tighter than twice the LP-relaxation bound of PM. We also show a similar result between PM and SM.

For ease of analysis, we introduce another model for the 2DK. Let us define $\mathcal{R}_0$ as follows:

$$\mathcal{R}_0 = \left\{ \boldsymbol{x} \in [0,1]^{|S_w|} \, : \, \sum_{j \in N} \sum_{s \in S_w^j} \bar{h}_j x_s \le H \right\}.$$

We remark that $\bar{h}_j$ is the height of the strip corresponding to the width pattern $\bar{\boldsymbol{a}}^s$ for each $j \in N$ and $s \in S_w^j$. Since $\mathcal{R}_0$ definition has only one constraint equivalent to constraint (15) in PE, $\mathcal{P}_{\text{PE}} \subseteq \mathcal{R}_0$. Accordingly, any $\boldsymbol{x}^{\text{PE}} \in \mathcal{P}_{\text{PE}}$ can be represented as a convex combination of the extreme points of $\mathcal{R}_0$.

Let $\bar{P}_h = \mathcal{R}_0 \cap \{0,1\}^{|S_w|} \setminus \{\boldsymbol{0}\}$. We denote each element of $\bar{P}_h$ as $\bar{\boldsymbol{b}}^t \in \{0,1\}^{|S_w|}$ for each $t \in S_h$ where $S_h = \{1, \dots, |\bar{P}_h|\}$. Then, each $\bar{\boldsymbol{b}}^t$ represents the usage of width patterns in $\bar{P}_w$ such that the total height is less than or equal to H. Each element of $\bar{P}_h$ can be matched to some element of P_h through function $g \, : \, \bar{P}_h \to P_h$ defined as

$$g(\bar{\boldsymbol{b}}^t)_i = \sum_{q \in Q_w^i} \sum_{s \in S_w(q)} \bar{b}_s^t, \quad \forall i \in M.$$

We note that g is an onto function because both $\bar{P}_h$ and P_h do not care about demand constraints. Subsequently, we define $S_h(r) = \{t \in S_h \, : \, g(\bar{\boldsymbol{b}}^t) = \boldsymbol{b}^r\}$ for each $r \in Q_h$. We note that $S_h = \cup_{r \in Q_h} S_h(r)$.

By utilizing $\bar{P}_h$ and $\bar{\boldsymbol{b}}^t$, $t \in S_h$, PE can be reformulated as follows, which we call PR:

$$\text{PR:} \quad \text{maximize} \quad \sum_{s \in S_w} \sum_{j \in N} \sum_{t \in S_h} \bar{p}_j \bar{a}_j^s \bar{b}_s^t x_t$$

$$\text{subject to} \quad \sum_{s \in S_w} \sum_{t \in S_h} \bar{a}_j^s \bar{b}_s^t x_t \le 1, \quad \forall j \in N, \tag{25}$$

$$\sum_{t \in S_h} x_t \leq 1, \tag{26}$$

$$x_t \in \{0, 1\}, \quad \forall t \in S_h.$$

Here, the variable x_t for each $t \in S_h$ represents whether to use width patterns used in $\bar{\boldsymbol{b}}^t$ or not. Note that constraint (26) means that only one $\bar{\boldsymbol{b}}^t$ can be chosen, and so $x_s = \sum_{t \in S_h} \bar{b}^t_s x_t$ which explains how PR is derived from PE for which constraint (15) is automatically satisfied due to the definitions of $\mathcal{R}_0$ and $\bar{P}_h$. We obtain the following result from the relationship between P_h and $\bar{P}_h$.

Proposition 6 $z^{\mathrm{PR}}_{\mathrm{LP}} \leq z^{\mathrm{SM}}_{\mathrm{LP}}$.

Proof Let $\boldsymbol{x}^{\mathrm{PR}} \in \mathcal{P}_{\mathrm{PR}}$ be an optimal solution of PR, where the corresponding objective value is $z^{\mathrm{PR}}_{\mathrm{LP}}$. We show that a feasible solution for the LP-relaxation of SM can be constructed from $\boldsymbol{x}^{\mathrm{PR}}$, which yields the same objective value with $z^{\mathrm{PR}}_{\mathrm{LP}}$.

Let us define $(\boldsymbol{x}^{\mathrm{SM}}, \boldsymbol{y}^{\mathrm{SM}}) \in \mathbb{R}^{|Q_w| \times |Q_h|}$ as

$$x^{\mathrm{SM}}_q = \sum_{s \in S_w(q)} \sum_{t \in S_h} \bar{b}^t_s x^{\mathrm{PR}}_t, \quad \forall q \in Q_w,$$

and

$$y^{\mathrm{SM}}_r = \sum_{t \in S_h(r)} x^{\mathrm{PR}}_t, \quad \forall r \in Q_h.$$

Then, $\boldsymbol{y}^{\mathrm{SM}}$ satisfies constraint (13) because

$$\sum_{r \in Q_h} y^{\mathrm{SM}}_r = \sum_{r \in Q_h} \sum_{t \in S_h(r)} x^{\mathrm{PR}}_t = \sum_{t \in S_h} x^{\mathrm{PR}}_t \leq 1,$$

where the last inequality holds due to constraint (26). This result also implies that $y^{\mathrm{SM}}_r \leq 1$ for each $r \in Q_h$. Next, we show that $\boldsymbol{x}^{\mathrm{SM}}$ satisfies constraints (9). For each $i \in M$, the following inequality

$$\sum_{j \in N_i} \sum_{s \in S_w} \sum_{t \in S_h} \bar{a}^s_j \bar{b}^t_s x^{\mathrm{PR}}_t \leq d_i \tag{27}$$

can be obtained by aggregating constraints (25) corresponding to $j \in N_i$. Then, we can see that

$$\begin{aligned} \sum_{q \in Q_w} a^q_i x^{\mathrm{SM}}_q &= \sum_{q \in Q_w} \sum_{s \in S_w(q)} \sum_{t \in S_h} a^q_i \bar{b}^t_s x^{\mathrm{PR}}_t = \sum_{q \in Q_w} \sum_{s \in S_w(q)} \sum_{t \in S_h} \left(\sum_{j \in N_i} \bar{a}^s_j \right) \bar{b}^t_s x^{\mathrm{PR}}_t \\ &= \sum_{j \in N_i} \sum_{s \in S_w} \sum_{t \in S_h} \bar{a}^s_j \bar{b}^t_s x^{\mathrm{PR}}_t \leq d_i, \end{aligned}$$

where the last inequality holds due to (27). This result implies that constraints (9) are satisfied by $\boldsymbol{x}^{\mathrm{SM}}$. Lastly, $(\boldsymbol{x}^{\mathrm{SM}}, \boldsymbol{y}^{\mathrm{SM}})$ also satisfies constraints (12) because, for each $i \in M$,

$$\sum_{r\in Q_h} b_i^r y_r^{\mathrm{SM}} = \sum_{r\in Q_h}\sum_{t\in S_h(r)} b_i^r x_t^{\mathrm{PR}} = \sum_{r\in Q_h}\sum_{t\in S_h(r)}\left(\sum_{q\in Q_w^i}\sum_{s\in S_w(q)} \bar{b}_s^t\right) x_t^{\mathrm{PR}}$$
$$= \sum_{t\in S_h}\sum_{q\in Q_w^i}\sum_{s\in S_w(q)} \bar{b}_s^t x_t^{\mathrm{PR}} = \sum_{q\in Q_w^i} x_q^{\mathrm{SM}}$$

by the definition of $\boldsymbol{x}^{\mathrm{SM}}$ and $\boldsymbol{y}^{\mathrm{SM}}$, as well as of $S_h(r)$ that allows replacing b_i^r by $g(\bar{\boldsymbol{b}}^t)_i$ in the second equality. Therefore, $(\boldsymbol{x}^{\mathrm{SM}}, \boldsymbol{y}^{\mathrm{SM}}) \in \mathcal{P}_{\mathrm{SM}}$.

Now, we show that the objective value corresponding to $(\boldsymbol{x}^{\mathrm{SM}}, \boldsymbol{y}^{\mathrm{SM}})$ is equivalent to $z_{\mathrm{LP}}^{\mathrm{PR}}$. By the definition of $S_w(q)$, the following equalities hold:

$$p_i a_i^q = \sum_{j\in N_i} \bar{p}_j \bar{a}_j^s, \quad \forall i \in M,\ \forall q \in Q_w,\ \forall s \in S_w(q).$$

Using these equalities and the definition of $\boldsymbol{x}^{\mathrm{SM}}$, the objective value corresponding to $(\boldsymbol{x}^{\mathrm{SM}}, \boldsymbol{y}^{\mathrm{SM}})$ can be represented as

$$\sum_{q\in Q_w}\sum_{i\in M} p_i a_i^q x_q^{\mathrm{SM}} = \sum_{q\in Q_w}\sum_{i\in M} p_i a_i^q\left(\sum_{s\in S_w(q)}\sum_{t\in S_h} \bar{b}_s^t x_t^{\mathrm{PR}}\right)$$
$$= \sum_{q\in Q_w}\sum_{s\in S_w(q)}\sum_{t\in S_h}\sum_{i\in M}\sum_{j\in N_i} \bar{p}_j \bar{a}_j^s \bar{b}_s^t x_t^{\mathrm{PR}}$$
$$= \sum_{s\in S_w}\sum_{t\in S_h}\sum_{j\in N} \bar{p}_j \bar{a}_j^s \bar{b}_s^t x_t^{\mathrm{PR}},$$

where the last term is equivalent to $z_{\mathrm{LP}}^{\mathrm{PR}}$. This result implies that $z_{\mathrm{LP}}^{\mathrm{PR}} \le z_{\mathrm{LP}}^{\mathrm{SM}}$. □

We note that there exists a 2DK instance where $z_{\mathrm{LP}}^{\mathrm{PR}} < z_{\mathrm{LP}}^{\mathrm{SM}}$, as shown in Appendix C. Although PR provides a tighter LP relaxation bound than SM, PR has excessively many variables by the definition of $\bar{\boldsymbol{b}}^t$'s, even far more than SM. In addition, some of $\bar{\boldsymbol{b}}^t$'s themselves represent feasible solutions for the 2DK. This implies that, even though the column generation approach can be applied to solve the LP relaxation of PR, the subproblems may be as difficult as solving the 2DK directly. Therefore, it is not recommended to utilize PR in practice.

Now, let $\boldsymbol{v}_0^i \in [0,1]^{|S_w|}$ for all $i \in V_0$ be the non-zero extreme points of $\mathcal{R}_0$, where V_0 denotes the index set of them and $v_{0\,s}^i$ for each $s \in S_w$ denotes each element of $\boldsymbol{v}_0^i$. We note that $\mathcal{R}_0$ is represented as the LP-relaxation of the feasible solution set of a binary knapsack problem. Hence, in analogy with extreme points of $\mathcal{R}'_k$ from the proof of Proposition 4, each $\boldsymbol{v}_0^i$ has at most one fractional component. Additionally, each $\boldsymbol{v}_0^i$ has at least one component whose value is 1 since $\bar{h}_j \le H$ for each $j \in N$. Let $\lfloor \boldsymbol{v}_0^i \rfloor = (\lfloor v_{01}^i \rfloor, \ldots, \lfloor v_{0\,|S_h|}^i \rfloor)$ and $\lceil \boldsymbol{v}_0^i \rceil = (\lceil v_{01}^i \rceil, \ldots, \lceil v_{0\,|S_h|}^i \rceil)$ for each $i \in V_0$. Then, each $\lfloor \boldsymbol{v}_0^i \rfloor$ is non-zero and corresponds to $\bar{\boldsymbol{b}}^t$ of $\bar{P}_h$ for some $t \in S_h$ by the definition of $\mathcal{R}_0$. On the other hand, each $\lceil \boldsymbol{v}_0^i \rceil - \lfloor \boldsymbol{v}_0^i \rfloor$ represents a zero vector or unit vector since $\boldsymbol{v}_0^i$ has at most one fractional component. Accordingly, if $\lceil \boldsymbol{v}_0^i \rceil - \lfloor \boldsymbol{v}_0^i \rfloor$ is a unit vector for some $i \in V_0$, then $\lceil \boldsymbol{v}_0^i \rceil - \lfloor \boldsymbol{v}_0^i \rfloor$ also corresponds to $\bar{\boldsymbol{b}}^t$ of $\bar{P}_h$ for some $t \in S_h$. Based on this relationship between $\boldsymbol{v}_0^i$'s and $\bar{\boldsymbol{b}}^t$'s with Proposition 6, we obtain an upper bound on $z_{\mathrm{LP}}^{\mathrm{PE}}$ using $z_{\mathrm{LP}}^{\mathrm{SM}}$.

Proposition 7 $z_{\mathrm{LP}}^{\mathrm{PE}} \leq 2 z_{\mathrm{LP}}^{\mathrm{PR}}$.

Proof Let $\boldsymbol{x}^{\mathrm{PE}} \in \mathcal{P}_{\mathrm{PE}}$ be an optimal solution for the LP-relaxation of PE, whose objective value is $z_{\mathrm{LP}}^{\mathrm{PE}}$. In the similar manner with the proof of Proposition 5, we construct a feasible solution for the LP-relaxation of PR from $\boldsymbol{x}^{\mathrm{PE}}$, where the corresponding objective value is greater than or equal to $(1/2) z_{\mathrm{LP}}^{\mathrm{PE}}$.

Since $\mathcal{P}_{\mathrm{PE}} \subseteq \mathcal{R}_0$, $\boldsymbol{x}^{\mathrm{PE}}$ can be represented as

$$\boldsymbol{x}^{\mathrm{PE}} = \sum_{i \in V_0} \lambda_0^i \boldsymbol{v}_0^i = \sum_{i \in V_0} \lambda_0^i \left(\lfloor \boldsymbol{v}_0^i \rfloor + f_0^i (\lceil \boldsymbol{v}_0^i \rceil - \lfloor \boldsymbol{v}_0^i \rfloor) \right), \tag{28}$$

for some $\lambda_0^i \in [0,1]$ for each $i \in V_0$ such that $\sum_{i \in V_0} \lambda_0^i \leq 1$. The second equality holds since $\boldsymbol{v}_0^i$ for each $i \in V_0$ has at most one fractional component, where $f_0^i = \sum_{s \in S_w} (v_{0s}^i - \lfloor v_{0s}^i \rfloor)$ for each $i \in V_0$. We note that $f_0^i < 1$ for each $i \in V_0$.

We partition V_0 into V_0^D and V_0^U where

$$V_0^D = \left\{ i \in V_0 \,:\, \sum_{s \in S_w} \sum_{j \in N} \bar{p}_j \bar{a}_j^s \lfloor v_{0s}^i \rfloor \geq \sum_{s \in S_w} \sum_{j \in N} \bar{p}_j \bar{a}_j^s f_0^i (\lceil v_{0s}^i \rceil - \lfloor v_{0s}^i \rfloor) \right\},$$

and

$$V_0^U = \left\{ i \in V_0 \,:\, \sum_{s \in S_w} \sum_{j \in N} \bar{p}_j \bar{a}_j^s \lfloor v_{0s}^i \rfloor < \sum_{s \in S_w} \sum_{j \in N} \bar{p}_j \bar{a}_j^s f_0^i (\lceil v_{0s}^i \rceil - \lfloor v_{0s}^i \rfloor) \right\}.$$

Then, $\lceil \boldsymbol{v}_0^i \rceil - \lfloor \boldsymbol{v}_0^i \rfloor$ for each $i \in V_0^U$ corresponds to $\bar{\boldsymbol{b}}^t$ for some $t \in S_h$, where $\bar{\boldsymbol{b}}^t$ represents a unit vector, since $\lceil \boldsymbol{v}_0^i \rceil - \lfloor \boldsymbol{v}_0^i \rfloor \neq \boldsymbol{0}$ by the definition of V_0^U. Also, $\lfloor \boldsymbol{v}_0^i \rfloor$ for each $i \in V_0^D$ corresponds to $\bar{\boldsymbol{b}}^t$ for some $t \in S_h$ as mentioned earlier. Let us define $D_0(t) = \{ i \in V_0^D \,:\, \lfloor \boldsymbol{v}_0^i \rfloor = \bar{\boldsymbol{b}}^t \}$ and $U_0(t) = \{ i \in V_0^U \,:\, \lceil \boldsymbol{v}_0^i \rceil - \lfloor \boldsymbol{v}_0^i \rfloor = \bar{\boldsymbol{b}}^t \}$ for each $t \in S_h$. Then, it is clear that V_0^D and V_0^U are partitioned into $D_0(t)$'s and $U_0(t)$'s, respectively.

Now, we define $\boldsymbol{x}^{\mathrm{PR}} \in \mathbb{R}_+^{|S_h|}$ as, for each $t \in S_h$,

$$x_t^{\mathrm{PR}} = \sum_{i \in D_0(t)} \lambda_0^i + \sum_{i \in U_0(t)} \lambda_0^i f_0^i,$$

where $x_t^{\mathrm{PR}} = 0$ if $D_0(t) = \emptyset$ and $U_0(t) = \emptyset$. We first show that $\boldsymbol{x}^{\mathrm{PR}} \in \mathcal{P}_{\mathrm{PR}}$. From the definition of $\boldsymbol{x}^{\mathrm{PR}}$, the following inequalities

$$\sum_{t\in S_h} x_t^{\mathrm{PR}} = \sum_{t\in S_h}\left(\sum_{i\in D_0(t)}\lambda_0^i + \sum_{i\in U_0(t)}\lambda_0^i f_0^i\right) \le \sum_{t\in S_h}\sum_{i\in D_0(t)}\lambda_0^i + \sum_{t\in S_h}\sum_{i\in U_0(t)}\lambda_0^i$$
$$= \sum_{i\in V_0^D}\lambda_0^i + \sum_{i\in V_0^U}\lambda_0^i = \sum_{i\in V_0}\lambda_0^i \le 1$$

hold. We note that the first inequality, which is valid because $f_0^i < 1$, becomes equality when $U_0(t) = \emptyset$ for all $t \in S_h$. This result implies that $\boldsymbol{x}^{\mathrm{PR}} \in [0,1]^{|S_h|}$ and $\boldsymbol{x}^{\mathrm{PR}}$ satisfies constraint (26). On the other hand, recall that, for each $t \in S_h$, $\lfloor \boldsymbol{v}_0^i \rfloor = \bar{\boldsymbol{b}}^t$ for each $i \in D_0(t)$ and $\lceil \boldsymbol{v}_0^i \rceil - \lfloor \boldsymbol{v}_0^i \rfloor = \bar{\boldsymbol{b}}^t$ for each $i \in U_0(t)$. From this relationship between $\bar{\boldsymbol{b}}^t$'s and $\boldsymbol{v}_0^i$'s, we have the following inequalities for each $s \in S_w$:

$$\begin{aligned}
\sum_{t\in S_h}\bar{b}_s^t x_t^{\mathrm{PR}} &= \sum_{t\in S_h}\left(\sum_{i\in D_0(t)}\lambda_0^i \bar{b}_s^t + \sum_{i\in U_0(t)}\lambda_0^i f_0^i \bar{b}_s^t\right)\\
&= \sum_{t\in S_h}\sum_{i\in D_0(t)}\lambda_0^i \lfloor v_{0s}^i \rfloor + \sum_{t\in S_h}\sum_{i\in U_0(t)}\lambda_0^i f_0^i (\lceil v_{0s}^i \rceil - \lfloor v_{0s}^i \rfloor)\\
&= \sum_{i\in V_0^D}\lambda_0^i \lfloor v_{0s}^i \rfloor + \sum_{i\in V_0^U}\lambda_0^i f_0^i (\lceil v_{0s}^i \rceil - \lfloor v_{0s}^i \rfloor)\\
&\le \sum_{i\in V_0^D}\lambda_0^i v_{0s}^i + \sum_{i\in V_0^U}\lambda_0^i v_{0s}^i = \sum_{i\in V_0}\lambda_0^i v_{0s}^i,
\end{aligned} \tag{29}$$

where the last inequality holds since $\boldsymbol{v}_0^i = \lfloor \boldsymbol{v}_0^i \rfloor + f_0^i(\lceil \boldsymbol{v}_0^i \rceil - \lfloor \boldsymbol{v}_0^i \rfloor)$ for each $i \in V_0$. Then, $\boldsymbol{x}^{\mathrm{PR}}$ satisfies constraints (25) because, for each $j \in N$,

$$\sum_{s\in S_w}\sum_{t\in S_h}\bar{a}_j^s \bar{b}_s^t x_t^{\mathrm{PR}} = \sum_{s\in S_w}\bar{a}_j^s \sum_{t\in S_h}\bar{b}_s^t x_t^{\mathrm{PR}} \le \sum_{s\in S_w}\bar{a}_j^s \sum_{i\in V_0}\lambda_0^i v_{0s}^i = \sum_{s\in S_w}\bar{a}_j^s x_s^{\mathrm{PE}} \le 1,$$

where the first inequality holds by (29) and the last inequality holds due to constraints (14). Therefore, $\boldsymbol{x}^{\mathrm{PR}} \in \mathcal{P}_{\mathrm{PR}}$.

The objective value corresponding to $\boldsymbol{x}^{\mathrm{PR}}$ can be represented as

$$\begin{aligned}
\sum_{s\in S_w}\sum_{j\in N}\sum_{t\in S_h}\bar{p}_j \bar{a}_j^s \bar{b}_s^t x_t^{\mathrm{PR}} &= \sum_{s\in S_w}\sum_{j\in N}\bar{p}_j \bar{a}_j^s \sum_{t\in S_h}\bar{b}_s^t x_t^{\mathrm{PR}}\\
&= \sum_{i\in V_0^D}\lambda_0^i\left(\sum_{s\in S_w}\sum_{j\in N}\bar{p}_j \bar{a}_j^s \lfloor v_{0s}^i \rfloor\right)\\
&\quad + \sum_{i\in V_0^U}\lambda_0^i\left(\sum_{s\in S_w}\sum_{j\in N}\bar{p}_j \bar{a}_j^s f_0^i (\lceil v_{0s}^i \rceil - \lfloor v_{0s}^i \rfloor)\right)\\
&= \sum_{i\in V_0}\lambda_0^i \max\\
&\quad \left\{\sum_{s\in S_w}\sum_{j\in N}\bar{p}_j \bar{a}_j^s \lfloor v_{0s}^i \rfloor, \sum_{s\in S_w}\sum_{j\in N}\bar{p}_j \bar{a}_j^s f_0^i (\lceil v_{0s}^i \rceil - \lfloor v_{0s}^i \rfloor)\right\},
\end{aligned}$$

where the second equality holds due to the equalities in (29). Here, the third equality holds due to the definition of V_0^D and V_0^U. On the other hand, $z_{\text{LP}}^{\text{PE}}$ can be represented as

$$
\begin{aligned}
z_{\text{LP}}^{\text{PE}} &= \sum_{s\in S_w}\sum_{j\in N}\bar{p}_j\bar{a}_j^s x_s^{\text{PE}} = \sum_{s\in S_w}\sum_{j\in N}\bar{p}_j\bar{a}_j^s\left(\sum_{i\in V_0}\lambda_0^i v_{0s}^i\right)\\
&= \sum_{i\in V_0}\lambda_0^i\left(\sum_{s\in S_w}\sum_{j\in N}\bar{p}_j\bar{a}_j^s\lfloor v_{0s}^i\rfloor + \sum_{s\in S_w}\sum_{j\in N}\bar{p}_j\bar{a}_j^s f_0^i(\lceil v_{0s}^i\rceil - \lfloor v_{0s}^i\rfloor)\right),
\end{aligned}
$$

where the last equality holds due to the equalities (28). Therefore, the objective value corresponding to $\boldsymbol{x}^{\text{PR}}$ is greater than or equal to $(1/2)z_{\text{LP}}^{\text{PE}}$. This result implies that $z_{\text{LP}}^{\text{PE}} \le 2z_{\text{LP}}^{\text{PR}}$. □

Because $z_{\text{LP}}^{\text{PM}} = z_{\text{LP}}^{\text{PE}}$ and $z_{\text{LP}}^{\text{PR}} \le z_{\text{LP}}^{\text{SM}}$ by Propositions 3 and 6, respectively, Proposition 7 implies that $z_{\text{LP}}^{\text{PM}} \le 2z_{\text{LP}}^{\text{SM}}$. Our findings throughout this paper can be summarized as the following theorem.

Theorem 8 $z^* \le z_{\text{LP}}^{\text{SM}} \le z_{\text{LP}}^{\text{PM}} \le z_{\text{LP}}^{\text{ML}} \le 2z_{\text{LP}}^{\text{PM}} \le 4z_{\text{LP}}^{\text{SM}}$.

Proof Propositions 2, 3, and 5 imply that $z_{\text{LP}}^{\text{PM}} \le z_{\text{LP}}^{\text{ML}} \le 2z_{\text{LP}}^{\text{PM}}$. Furthermore, by Propositions 3, 6 and 7, we have

$$2z_{\text{LP}}^{\text{PM}} = 2z_{\text{LP}}^{\text{PE}} \le 4z_{\text{LP}}^{\text{PR}} \le 4z_{\text{LP}}^{\text{SM}}.$$

Therefore, with Proposition 1, the result follows. □

We now introduce a tight example for the relationship $z_{\text{LP}}^{\text{SM}} \le z_{\text{LP}}^{\text{PM}} \le z_{\text{LP}}^{\text{ML}}$.

Example 2 The large plate has a (width, height) pair of (1, 2) and there is only one item with (width, height)=(1, 1), which has a unit profit and unit demand. We note that this 2DK instance is constrained by definition. Recall that z^* is the optimal objective value of this 2DK instance. It is clear that $z^* = 1$ since this item can be cut from this plate. Let us consider ML for this example. By definition, ML has only one variable $x_{11} \in [0, 1]$. Let $x_{11}^{\text{ML}} = 1$. It is trivial that x_{11}^{ML} is an optimal solution for the LP-relaxation of ML. Accordingly, $z_{11}^{\text{ML}} = 1$. From Propositions 1 and 5, we can see that

$$1 = z^* \le z_{\text{LP}}^{\text{SM}} \le z_{\text{LP}}^{\text{PM}} \le z_{\text{LP}}^{\text{ML}} = 1,$$

for this instance, that is, $z_{\text{LP}}^{\text{SM}} = z_{\text{LP}}^{\text{PM}} = z_{\text{LP}}^{\text{ML}}$. This instance is also a strict example for the relationship $z_{\text{LP}}^{\text{ML}} \le 2z_{\text{LP}}^{\text{PM}} \le 4z_{\text{LP}}^{\text{SM}}$.

Additionally, we present an asymptotically tight example for the relationship $z_{\text{LP}}^{\text{ML}} \le 2z_{\text{LP}}^{\text{PM}} \le 4z_{\text{LP}}^{\text{SM}}$ as follows.

Example 3 The large plate has a (width, height) pair of $(2M, 2M)$ and there are four different types of items sharing the same (width, height) pair, $(M+1, M+1)$, with unit profit and unit demand. Here, only one item can be cut from this plate. Now, let us consider ML for this example, that is, $z^* = 1$. We define $\boldsymbol{x}^{\mathrm{ML}} \in \mathbb{R}_+^{n\times n}$ as follows:

$$x_{11}^{\mathrm{ML}} = x_{33}^{\mathrm{ML}} = x_{21}^{\mathrm{ML}} = x_{43}^{\mathrm{ML}} = \frac{M}{M+1};\ x_{\mathrm{jk}}^{\mathrm{ML}} = 0, \text{ otherwise.}$$

It can be easily shown that $\boldsymbol{x}^{\mathrm{ML}} \in \mathcal{P}_{\mathrm{ML}}$. Also, the corresponding objective value is $4M/(M+1)$, and it converges to 4 as M goes to infinity. On the other hand, we can see that 4 is a trivial upper bound on $z_{\mathrm{LP}}^{\mathrm{ML}}$. Therefore, $\boldsymbol{x}^{\mathrm{ML}}$ is an optimal solution for the LP-relaxation of ML as $M \to \infty$.

Possible width patterns and height patterns are given as follows:

$$P_w = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\},$$
$$P_h = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\},$$

respectively. Let us consider PM for this example. Then, we can see that constraint (10) reduces to $\sum_{q\in Q_w} x_q \le 2M/(M+1)$, where the left hand-side is equivalent to the objective value. Hence, $z_{\mathrm{LP}}^{\mathrm{PM}}$ is less than or equal to $2M/(M+1)$. We define $\boldsymbol{x}^{\mathrm{PM}} \in \mathcal{P}_{\mathrm{PM}}$ as $x_1^{\mathrm{PM}} = x_2^{\mathrm{PM}} = M/(M+1)$ and $x_3^{\mathrm{PM}} = x_4^{\mathrm{PM}} = 0$. Since the corresponding objective value is $2M/(M+1)$, $\boldsymbol{x}^{\mathrm{PM}}$ is an optimal solution. Therefore, $z_{\mathrm{LP}}^{\mathrm{PM}}$ converges to 2 as M goes to infinity.

Now, let us consider SM for this example. The aggregation of constraints (12) results in $\sum_{q\in Q_w} x_q \le 1$ by constraint (13), which implies that $z_{\mathrm{LP}}^{\mathrm{SM}} \le 1$. We define $(\boldsymbol{x}^{\mathrm{SM}}, \boldsymbol{y}^{\mathrm{SM}}) \in \mathbb{R}_+^{|Q_w|\times|Q_h|}$ as $x_1^{\mathrm{SM}} = y_1^{\mathrm{SM}} = 1$ and otherwise 0. It is easy to check that $(\boldsymbol{x}^{\mathrm{SM}}, \boldsymbol{y}^{\mathrm{SM}}) \in \mathcal{P}_{\mathrm{SM}}$. The corresponding objective value is 1, that is, $(\boldsymbol{x}^{\mathrm{SM}}, \boldsymbol{y}^{\mathrm{SM}})$ is an optimal solution for the LP-relaxation of SM because $z_{\mathrm{LP}}^{\mathrm{SM}} \le 1$. Therefore, this example satisfies $z_{\mathrm{LP}}^{\mathrm{ML}} \le 2z_{\mathrm{LP}}^{\mathrm{PM}} \le 4z_{\mathrm{LP}}^{\mathrm{SM}}$ tightly as M goes to infinity. Furthermore, this instance for any given $M \ge 3$ is a strict example for the relationship $z_{\mathrm{LP}}^{\mathrm{SM}} \le z_{\mathrm{LP}}^{\mathrm{PM}} \le z_{\mathrm{LP}}^{\mathrm{ML}}$.

We note that the example contains items with equivalent width, height, and profit, but not merged in a single type ($m = 1$). However, the same result is achievable by modifying the widths and heights of items differently from each other, where the ratio between the height and width for each item converges to 1 when M goes to ∞, while item widths and heights remain greater than M.

4 Conclusion

This paper presents several integer linear programming models for the 2DK based on pattern-based models for the 2DCS. In addition, the well-known level packing model for the 2DK is modified by adding some valid inequalities, enhancing its LP-relaxation bound and making its structure easier to analyze. Then, we compare the LP-relaxation bounds of these models. The results show that the level packing model provides weaker LP-relaxation bounds compared to pattern-based models. We also investigate the worst-case ratio between the LP-relaxation bounds of the level packing model and pattern-based models. The ratio between the level packing model and strip packing model is 2, while the computational results of Appendix B and Lodi and Monaci [13] observed that the ratios are relatively close to 1 for the benchmark test instances. The ratio further increases when compared with the staged pattern model. Therefore, more elaborate computational comparisons should be conducted with various instances for the level packing model and pattern-based models. For future works, the presented models can be compared to other models derived from 2DCS models, such as the arc-flow model [14] and the one-cut model [17]. Including the relationship of LP-relaxation bounds, analyzing properties of various models will provide useful information to devise more efficient exact and heuristic algorithms for the 2DK.

Appendix A: LP-relaxations of SM and $\mathrm{SM}^{=}$

Proposition 9 $z_{\mathrm{LP}}^{\mathrm{SM}} = z_{\mathrm{LP}}^{\mathrm{SM}^{=}}$.

Proof We note that both SM and $\mathrm{SM}^{=}$ consider not just maximal height patterns, but all height patterns. It is clear that $z_{\mathrm{LP}}^{\mathrm{SM}} \geq z_{\mathrm{LP}}^{\mathrm{SM}^{=}}$ since feasible solutions for the LP-relaxation of $\mathrm{SM}^{=}$ are also feasible for the LP-relaxation of SM. Hence, we only prove that $z_{\mathrm{LP}}^{\mathrm{SM}} \leq z_{\mathrm{LP}}^{\mathrm{SM}^{=}}$. For any given $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{P}_{\mathrm{SM}}$, we show that a feasible solution for the LP-relaxation of $\mathrm{SM}^{=}$ can be constructed, which yields the same objective value as $z_{\mathrm{LP}}^{\mathrm{SM}}$.

Let $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ be a feasible solution for the LP-relaxation of SM. We define $\xi_i = \sum_{q \in Q_w^i} \hat{x}_q$, $\delta_i = \sum_{r \in Q_h} b_i^r \hat{y}_r - \xi_i$ for each $i \in M$, and $\Delta = (\xi_1 + \delta_1, \dots, \xi_m + \delta_m)$. Note that $\delta_i \geq 0$ for each $i \in M$ since $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \in \mathcal{P}_{\mathrm{SM}}$. If $\sum_{i \in M} \delta_i = 0$, then it is clear that $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \in \mathcal{P}_{\mathrm{SM}^{=}}$. Hence, assume that $\sum_{i \in M} \delta_i > 0$. Let $\mathcal{C}$ be the convex hull of $\mathbf{0}$ and all the height patterns ($\boldsymbol{b}^r$'s). Then, it is clear that $\Delta = \sum_{r \in Q_h} \boldsymbol{b}^r \hat{y}_r \in \mathcal{C}$. Additionally, let $k = \arg\min_{i \in M} \{\delta_i > 0\}$, and let $\bar{\Delta}$ be the same vector as Δ except the kth component is replaced with ξ_k. We first show that $\bar{\Delta} \in \mathcal{C}$, that is, there exists $\bar{\boldsymbol{y}} \in \mathbb{R}_+^{|Q_h|}$ such that $\bar{\Delta} = \sum_{r \in Q_h} \boldsymbol{b}^r \bar{y}_r$ and $\sum_{r \in Q_h} \bar{y}_r \leq 1$. Let Δ_0 be the same vector as Δ except the kth component replaced with 0. For each $r \in Q_h$, let us define $\hat{\boldsymbol{b}}^r$ be the same vector as $\boldsymbol{b}^r$ with the kth component replaced with 0. Then, Δ_0 is equal to $\sum_{r \in Q_h} \hat{\boldsymbol{b}}^r \hat{y}_r$ where $\sum_{r \in Q_h} \hat{y}_r \leq 1$, which means that $\Delta_0 \in \mathcal{C}$ because $\hat{\boldsymbol{b}}^r \in \mathcal{C}$ for each $r \in Q_h$ by the definition of the height pattern. On the other hand, $\bar{\Delta}$ can be represented as a convex combination of Δ and Δ_0 as follows:

$$\bar{\Delta} = \frac{\xi_k}{\xi_k + \delta_k} \Delta + \frac{\delta_k}{\xi_k + \delta_k} \Delta_0.$$

Therefore, $\bar{\Delta} \in \mathcal{C}$ since $\Delta \in \mathcal{C}$ and $\Delta_0 \in \mathcal{C}$, which implies that there exists $\bar{\boldsymbol{y}} \in \mathbb{R}_+^{|Q_h|}$ such that $\bar{\Delta} = \sum_{r \in Q_h} \boldsymbol{b}^r \bar{y}_r$ and $\sum_{r \in Q_h} \bar{y}_r \leq 1$.

We iterate the above procedure, replacing further components of $\bar{\Delta}$, so obtaining an updated vector $\bar{\boldsymbol{y}}$, until we reach $\bar{\Delta} = (\xi_1, \dots, \xi_m)$. The resulting $\bar{\boldsymbol{y}} \in \mathbb{R}_+^{|Q_h|}$ satisfies $\sum_{r \in Q_h} b_i^r \bar{y}_r = \xi_i = \sum_{q \in Q_w^i} \hat{x}_q$ for each $i \in M$ and $\sum_{r \in Q_h} \bar{y}_r \le 1$. Therefore, $(\hat{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \mathcal{P}_{\mathrm{SM}^=}$. It is clear that the corresponding objective value is the same as $z_{\mathrm{LP}}^{\mathrm{SM}}$ because it depends only on $\hat{\boldsymbol{x}}$. Therefore, the result follows. □

Appendix B: Computational results

In this section, we compare the different LP-relaxation models computationally. We report the LP-relaxation bound and computation time for each model: LM, ML, PM, and SM. The computational experiments were conducted on a CPU with Intel(R) Core(TM) i7-4770 and 16GB RAM using the solvers offered by Xpress 8.9 [20]. The models LM and ML were solved using the default solver of Xpress. On the other hand, the pattern-based models, PM and SM, were solved using the column generation method. The subproblems for generating columns are defined as the bounded knapsack problems, and each problem is solved using Xpress' default solver. We used 20 instances from Alvarez-Valdés et al. [1], and these instances are classified as large instances in previous literature. The results are given in Table 1. The optimal values z^* are obtained from Alvarez-Valdes et al. [2]. The columns z_{LP} and t_{LP} represent the LP-relaxation bound and the computation time in seconds, respectively. In addition, the problem size of each model is reported in Table 2. For LM and ML, we report the numbers of variables and constraints (Vars and Cons in Table 2). For the pattern-based models, the numbers of generated patterns are compared. The columns WP and HP represent the numbers of width and height patterns, respectively.

The LP-relaxation of LM showed the shortest solving time, attributed to its concise formulation. Besides, ML could yield decreased upper bounds for some instances; however, it incurs some overhead from the inclusion of additional inequalities. The difference in the number of constraints resulting in these results can be seen in Table 2. The LP-relaxation of PM obtained better bounds in a shorter time than ML. Although SM takes more time compared to PM, the LP-relaxation of SM gives the tightest bounds. As can be seen in Table 2, SM generated more width patterns than PM. SM obtains the optimal objective values in some instances.

Appendix C: Comparison between SM and PR

Let us consider the following 2DK instance: $I = (m, H, W, \mathbf{h}, \mathbf{w}, \mathbf{d}, \mathbf{p}) = (2,\ 15,\ 1,\ (7,\ 5),\ (1,\ 1),\ (1,\ 2),\ (7,\ 5))$. We note that SM considers not just maximal height patterns, but all height patterns. Hence, by definition, $P_w = \{(1,0),(0,1)\}$ and $P_h = \{(2,0),(1,1),(1,0),(0,3),(0,2),(0,1)\}$. Let us define $(\boldsymbol{x}^{\mathrm{SM}}, \boldsymbol{y}^{\mathrm{SM}}) \in \mathbb{R}^{|Q_w| \times |Q_h|}$ as $\boldsymbol{x}^{\mathrm{SM}} = (1, 1.5)$ and $\boldsymbol{y}^{\mathrm{SM}} = (0.5, 0, 0, 0.5, 0, 0)$. It can be easily checked that $(\boldsymbol{x}^{\mathrm{SM}}, \boldsymbol{y}^{\mathrm{SM}}) \in \mathcal{P}_{\mathrm{SM}}$ and the corresponding objective value is 14.5. Therefore, $z_{\mathrm{LP}}^{\mathrm{SM}}$ is greater than or equal to 14.5.

Table 1 Comparison of the LP-relaxations of the four models

Instance	z^*	LM		ML		PM		SM	
		z_{LP}	t_{LP}	z_{LP}	t_{LP}	z_{LP}	t_{LP}	z_{LP}	t_{LP}
ATP30	140,168	140,904.00	0.11	140,904.00	2.61	140,814.70	1.14	140,207.00	23.05
ATP31	820,260	825,184.00	0.25	825,184.00	6.01	824,220.00	1.14	820,868.50	74.50
ATP32	37,880	38,068.00	0.14	38,068.00	2.45	37,910.10	1.27	37,889.50	112.46
ATP33	235,580	236,903.00	0.14	236,903.00	4.81	235,734.00	6.77	235,580.00	35.28
ATP34	356,159	362,520.00	0.06	362,520.00	1.27	357,477.10	0.45	356,931.10	10.06
ATP35	614,429	623,040.00	0.09	623,040.00	1.64	617,352.80	0.53	616,651.40	12.95
ATP36	129,262	131,028.00	0.06	131,028.00	2.19	130,136.40	0.58	129,486.80	11.30
ATP37	384,478	387,640.00	0.16	387,640.00	1.88	385,900.00	1.42	384,665.30	56.00
ATP38	259,070	261,698.00	0.11	261,698.00	2.56	259,434.50	0.78	259,329.50	39.58
ATP39	266,135	269,538.00	0.05	269,538.00	1.19	268,668.00	0.53	266,585.50	19.56
ATP40	63,945	68,547.30	0.39	68,076.30	3.84	64,425.80	0.91	63,963.40	53.20
ATP41	202,305	215,993.00	0.13	213,954.80	0.91	205,389.20	0.45	202,305.00	17.87
ATP42	32,589	34,080.10	0.49	33,691.70	4.56	32,932.90	1.64	32,789.00	94.56
ATP43	208,998	222,175.70	0.25	221,279.00	1.31	214,503.60	1.11	212,093.30	64.20
ATP44	70,940	77,453.50	0.14	77,082.90	1.34	74,652.50	0.38	72,658.40	15.20
ATP45	74,205	77,892.40	0.11	77,484.80	1.55	74,324.90	0.27	74,205.00	8.84
ATP46	146,402	154,646.50	0.14	154,646.50	1.25	148,735.20	0.53	146,402.00	35.54
ATP47	144,317	157,521.80	0.16	157,160.30	0.73	150,603.00	0.45	144,526.50	23.75
ATP48	165,428	173,553.00	0.11	173,504.70	1.08	166,929.80	0.67	165,944.50	17.87
ATP49	206,965	226,610.40	0.06	224,695.20	0.55	210,651.60	0.44	208,511.50	7.30

Now, let us consider PR for this instance. The instance I can be transformed into the instance $\bar{I} = (3, 15, 1, (7, 5, 5), (1, 1, 1), (1, 1, 1), (7, 5, 5))$. Then,

$$\bar{P}_w = \{(1,0,0),(0,1,0),(0,0,1)\},$$

and

$$\bar{P}_h = \{(1,1,0),(1,0,1),(0,1,1),(1,0,0),(0,1,0),(0,0,1)\}.$$

We note that each component of each element of $\bar{P}_h$ indicates the usage of the corresponding width pattern. In the objective function of PR, we can see that

$$\sum_{s \in S_w} \sum_{j \in N} \bar{p}_j \bar{a}_j^s \bar{b}_s^t \le 12,$$

for each $t \in S_h$. From this observation, we can obtain an upper bound on $z_{\mathrm{LP}}^{\mathrm{PR}}$ as follows:

$$\sum_{s \in S_w} \sum_{j \in N} \sum_{t \in S_h} \bar{p}_j \bar{a}_j^s \bar{b}_s^t x_t \le \sum_{t \in S_h} 12 x_t \le 12,$$

where the last inequality holds due to constraint (26). Therefore, we can see that

Table 2 The problem size of the four models

Instance	LM		ML		PM	SM	
	Vars	Cons	Vars	Cons	WP	WP	HP
ATP30	18,528	655	18,528	821	78	111	113
ATP31	33,411	883	33,411	1086	102	136	159
ATP32	31,125	834	31,125	1002	111	172	234
ATP33	25,200	767	25,200	944	88	102	143
ATP34	8515	442	8515	592	55	83	79
ATP35	11,781	527	11,781	688	59	80	85
ATP36	11,781	529	11,781	692	56	64	71
ATP37	24,753	761	24,753	887	86	119	152
ATP38	20,503	689	20,503	836	80	116	139
ATP39	13,366	556	13,366	636	65	81	100
ATP40	42,195	994	42,195	1044	110	167	179
ATP41	15,753	602	15,753	654	73	92	128
ATP42	52,975	1127	52,975	1242	121	201	257
ATP43	33,670	892	33,670	923	98	148	184
ATP44	19,306	674	19,306	718	78	107	112
ATP45	12,246	527	12,246	625	65	101	87
ATP46	19,503	664	19,503	702	83	155	163
ATP47	20,910	693	20,910	728	86	141	134
ATP48	14,028	568	14,028	621	68	95	107
ATP49	7140	403	7140	459	52	70	67

$$z_{\mathrm{LP}}^{\mathrm{PR}} \leq 12 < 14.5 \leq z_{\mathrm{LP}}^{\mathrm{SM}},$$

for this instance.

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Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

References

1. Alvarez-Valdés, R., Parajón, A., et al.: A tabu search algorithm for large-scale guillotine (un) constrained two-dimensional cutting problems. Comput. Oper. Res. **29**(7), 925–947 (2002)
2. Alvarez-Valdes, R., Martí, R., Tamarit, J.M., et al.: Grasp and path relinking for the two-dimensional two-stage cutting-stock problem. INFORMS J. Comput. **19**(2), 261–272 (2007)
3. Becker, H., Martin, M., Araujo, O., et al.: Comparative analysis of mathematical formulations for the two-dimensional guillotine cutting problem. Int. Trans. Oper. Res. **31**(5), 3010–3035 (2023)
4. Belov, G., Scheithauer, G.: A branch-and-cut-and-price algorithm for one-dimensional stock cutting and two-dimensional two-stage cutting. Eur. J. Oper. Res. **171**(1), 85–106 (2006)
5. Caprara, A., Monaci, M.: On the two-dimensional knapsack problem. Oper. Res. Lett. **32**(1), 5–14 (2004)
6. Dyckhoff, H.: A new linear programming approach to the cutting stock problem. Oper. Res. **29**(6), 1092–1104 (1981)
7. Gilmore, P.C., Gomory, R.E.: Multistage cutting stock problems of two and more dimensions. Oper. Res. **13**(1), 94–120 (1965)
8. Hifi, M.: Exact algorithms for large-scale unconstrained two and three staged cutting problems. Comput. Optim. Appl. **18**(1), 63–88 (2001)
9. Hifi, M., M'Hallah, R.: Strip generation algorithms for constrained two-dimensional two-staged cutting problems. Eur. J. Oper. Res. **172**(2), 515–527 (2006)
10. Hifi, M., Roucairol, C.: Approximate and exact algorithms for constrained (un) weighted two-dimensional two-staged cutting stock problems. J. Combin. Optim. **5**(4), 465–494 (2001)
11. Iori, M., De Lima, V.L., Martello, S., et al.: Exact solution techniques for two-dimensional cutting and packing. Eur. J. Oper. Res. **289**(2), 399–415 (2021)
12. Kwon, S.J., Joung, S., Lee, K.: Comparative analysis of pattern-based models for the two-dimensional two-stage guillotine cutting stock problem. Comput. Oper. Res. **109**, 159–169 (2019)
13. Lodi, A., Monaci, M.: Integer linear programming models for 2-staged two-dimensional knapsack problems. Math. Program. **94**(2–3), 257–278 (2003)
14. Macedo, R., Alves, C., Valério de Carvalho, J.: Arc-flow model for the two-dimensional guillotine cutting stock problem. Comput. Oper. Res. **37**(6), 991–1001 (2010)
15. Mrad, M., Meftahi, I., Haouari, M.: A branch-and-price algorithm for the two-stage guillotine cutting stock problem. J. Oper. Res. Soc. **64**(5), 629–637 (2013)
16. Russo, M., Boccia, M., Sforza, A., et al.: Constrained two-dimensional guillotine cutting problem: upper-bound review and categorization. Int. Trans. Oper. Res. **27**(2), 794–834 (2020)
17. Silva, E., Alvelos, F., Valério de Carvalho, J.: An integer programming model for two-and three-stage two-dimensional cutting stock problems. Eur. J. Oper. Res. **205**(3), 699–708 (2010)
18. Suliman, S.M.: Pattern generating procedure for the cutting stock problem. Int. J. Prod. Econ. **74**(1–3), 293–301 (2001)

19. Valério de Carvalho, J.: Exact solution of bin-packing problems using column generation and branch-and-bound. Ann. Oper. Res. **86**, 629–659 (1999)
20. Xpress: Xpress-optimizer 8.9 (2020) . URL http://www.fico.com/en