International Statistical Review

International Statistical Review (2024) doi: 10.1111/insr.12589

New Scheme of Empirical Likelihood Method for Ranked Set Sampling: Applications to Two One-Sample Problems

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Summary

We propose a novel empirical likelihood (EL) approach for ranked set sampling (RSS) that leverages the ranking structure and information of the RSS. Our new proposal suggests constraining the sum of the within-stratum probabilities of each rank stratum to $1/H$, where H is the number of rank strata. The use of the additional constraints eliminates the need of subjective weight selection in unbalanced RSS and facilitates a seamless extension of the method for balanced RSS to unbalanced RSS. We apply our new proposal to testing one sample population mean and evaluate its performance through a numerical study and two real-world data sets, examining obesity from body fat data and symmetry of dental size from human tooth size data. We further consider the extension of the proposed EL method to jackknife EL.

Key words: balanced ranked set sampling; empirical likelihood; jackknife empirical likelihood one sample test; unbalanced ranked set sampling.

1 Introduction

Ranked set sampling (RSS) is a sampling technique that divides the population into strata based on the rank order of a sample. It is more cost-effective than simple random sampling in various applications, such as estimating and testing the mean, variance, quantiles, distribution function and area under the curve. Studies of the RSS population mean estimation can be found in Takahashi and Wakimoto (1968). The nonparametric maximum likelihood estimation of the cumulative distribution function using RSS is dealt with in Kvam and Samaniego (1994). Variance estimators using RSS were proposed in Stokes (1980), MacEachern et al. (2002) and Chen and Lim (2011). Ahn et al. (2014) introduced a population mean inference method using the Student's t approximation for pivotal statistics distribution. Wang et al. (2016) and Wang et al. (2017) developed inference procedures for continuous treatment effects in two-stage cluster randomised designs with both balanced and unbalanced RSS. We refer readers to Chen et al. (2006) and Wolfe (2012) for a detailed review of RSS.

On the other hand, the empirical likelihood (EL) method by Owen (1990) is a nonparametric method to construct a likelihood function using the empirical distribution of the data, rather than assuming a particular parametric distribution. It has been known to have several advantages over other methods, which include flexibility and robustness. It can be applied to a wide range of models of both continuous and discrete data and is model-free in the sense that it does not assume a specific form of distribution for the data. Further, it has been known to be particularly more powerful and efficient in testing hypotheses and interval estimation than other methods, when the data are from a non-normal distribution (Owen, 1990, 2001; Pen & Schick, 2013). A detailed review of the EL method can be found in the book by Owen (2001).

Unlike its success for simple random sampling (SRS), EL methods for RSS have been discussed by only a few authors. Baklizi (2009) studied the EL inference for the population mean and quantiles. Liu et al. (2009) considered the interval estimation and testing of the mean of one and two sample problems and general estimating equations. Both are methods for balanced RSS (BRSS). Later, Baklizi (2011) extended the quantile estimation in Baklizi (2009) to unbalanced RSS (URSS) and Moon et al. (2022) developed the EL method to estimate and test the area under the receiver operating characteristic curve for both BRSS and URSS. However, all these previous studies simply extended the EL methods for SRS into RSS without considering the rank structure of RSS. To be specific, in RSS, the sum of observation probabilities of each rank stratum is uniform and equal to $1/H$ where H is the size of a ranked set. However, previous studies did not consider this and impose the sum of all probabilities equal to 1 as in SRS. This strategy works fine for BRSS but is problematic for URSS. For this reason, Moon et al. (2022) suggested constraining the weighted sum of strata statistics or probabilities, where the weights are subjectively chosen depending on the sample size and the estimated variance of each rank stratum.

In this paper, we propose a new EL scheme for RSS and the EL test to test the population means that fully incorporates the ranking structure of the data and, as a consequence, can be applied to both BRSS and URSS. Our new method considers the ranking information more precisely and imposes additional constraints so that the sum of observation probabilities in each rank stratum is equal to $1/H$. This approach eliminates the need to choose the weights and can naturally be applied to both BRSS and URSS.

The remainder of the paper is organised as follows. In Section 2, we propose a new EL scheme for balanced/unbalanced RSS data and propose an EL test to test the population mean. We compute the asymptotic null distribution of the EL test statistic as the chi-square distribution with one degree of freedom. In Section 3, we review existing methods for testing the population mean and numerically compare the size and power of the new test with those of the existing methods reviewed. In Section 4, we apply the method to two real data examples, to testing the mean of the body mass index data and to testing the symmetry of the tooth size. In Section 5, we extend our EL method for the population mean to the jackknife EL (JEL) method. In Section 6, we conclude the paper with a discussion on extensions of the suggested scheme to other problems.

2 New EL Method for RSS

Suppose we have RSS data in the form of

$$\{(y_i, h_i, r_i),\ i = 1,\ 2,\ \cdots,\ n\},$$

where y_i has the rank h_i among r_i observations. Generally, we assume $r_i = H$, for every $i = 1, \cdots, n$, where H denotes the set size of RSS, so that $h_i \in \{1, \cdots, H\}$. Let $n_h = \sum_{i=1}^{H} I(h_i = h)$ be the number of units with rank h and so $n = \sum_{h=1}^{H} n_h$ is the total sample size of the RSS. For the same number of units $n_h = m$ for $h = 1, \cdots, H$, we obtain a balanced ranked set sample. The RSS data can also be written in the form of $Y_{[h],r}$ denoting the r-th observation having rank h for $h = 1, \cdots, H$ and $r = 1, \cdots, n_h$.

In this section, we propose a new EL method for one sample problem which is well-defined for both balanced and unbalanced RSS. Let $\mu_h = E(Y_i || r_i = h) = E(Y_{[h],r})$ and $p = (p_{1,1}, \cdots, p_{h,r}, \cdots, p_{H,n_H})$ be a probability vector such that $\sum_{h=1}^{H}\sum_{r=1}^{n_h} p_{h,r} = 1$ and $p_{h,r} \geq 0$ for $h = 1, \cdots, H$ and $r = 1, \cdots, n_h$. We define the EL of the population mean μ for RSS data as

$$\mathscr{L}(\mu) = \max\left\{\prod_{h=1}^{H}\prod_{r=1}^{n_h} p_{h,r} : \sum_{r=1}^{n_h} p_{h,r} = \frac{1}{H}, \sum_{h=1}^{H}\sum_{r=1}^{n_h} p_{h,r}\left(Y_{[h],r} - \mu_h\right) = 0\right\}. \tag{1}$$

The empirical likelihood in (1) makes the probability of each rank stratum be equal to $1/H$, which is different from the formulation in existing methods including Baklizi (2009), Baklizi (2011), Moon et al. (2022) and others. Baklizi (2009) only considered BRSS with $n_h = m$ for $h = 1, 2, \ldots, H$. Baklizi (2011) and Moon et al. (2022) further considered the EL for URSS but they all constrained the total sum or weighted sum of probabilities to one. To be specific, to take care of the unbalancedness in the sample design, they considered the strata weight w_h and constrained the weighted sum of the probabilities to one as

$$\sum_{h=1}^{H} w_h \sum_{r=1}^{n_h} p_{h,r} = 1, \tag{2}$$

and the weighted average of the strata means to the population mean μ as

$$\sum_{h=1}^{H} w_h \sum_{r=1}^{n_h} p_{h,r} Y_{[h],r} = \mu, \tag{3}$$

where $w_h = (Hn_h)^{-1}$ was subjectively chosen to depend on n_h, $h = 1, 2, \ldots, H$.

In this paper, we suggest to constrain the sum of the within-stratum probabilities to $1/H$ as

$$\sum_{r=1}^{n_h} p_{h,r} = \frac{1}{H}, \quad h = 1, 2, \ldots, H \tag{4}$$

instead of (2), which fits well with the basic principle of RSS. By assuming (4), the constraint on the population mean becomes straightforward as

$$\sum_{h=1}^{H}\sum_{r=1}^{n_h} p_{h,r} Y_{[h],r} = \frac{1}{H}\sum_{h=1}^{H}\left(H\sum_{r=1}^{n_h} p_{h,r} Y_{[h],r}\right) = \mu, \tag{5}$$

and equivalently

$$\sum_{h=1}^{H}\sum_{r=1}^{n_h} p_{h,r}\left(Y_{[h],r} - \mu\right) = \frac{1}{H}\sum_{h=1}^{H}\left\{H\sum_{r=1}^{n_h} p_{h,r}\left(Y_{[h],r} - \mu_h\right)\right\} = 0.$$

Thus, it eliminates the need for subjective selection of stratum weights w_h, $h = 1, 2, \ldots, H$.

To evaluate $\mathscr{L}(\mu)$, we maximize the Lagrangian function

$$f(\mu) = \sum_{h=1}^{H}\sum_{r=1}^{n_h} \log p_{h,r} - \sum_{h=1}^{H}\lambda_h\left(\sum_{r=1}^{n_h} p_{h,r} - \frac{1}{H}\right) - v\left(\sum_{h=1}^{H}\sum_{r=1}^{n_h} p_{h,r}\left(Y_{[h],r} - \mu_h\right)\right),$$

with respect to $p_{h,r}$, λ_h and v. The maximum of $f(\mu)$ is achieved when

$$p_{h,r} = \frac{1}{\lambda_h + v\left(Y_{[h],r} - \mu_h\right)} \tag{6}$$

where λ_h and v, the Lagrange multipliers, are the solutions of

$$\sum_{r=1}^{n_h} \frac{1}{\lambda_h + v\left(Y_{[h],r} - \mu_h\right)} = \frac{1}{H} \tag{7}$$

and

$$\sum_{h=1}^{H}\sum_{r=1}^{n_h} \frac{Y_{[h],r} - \mu_h}{\lambda_h + v\left(Y_{[h],r} - \mu_h\right)} = 0. \tag{8}$$

Now we study the procedure to test the hypothesis $H_0 : \mu = \mu_0$ using the above empirical log-likelihood ratio statistic, which incorporates the RSS design into the likelihood function. Note that without the mean constraint (5), the EL is maximised at $p_{h,r} = (Hn_h)^{-1}$ and so the EL ratio statistic is

$$\mathscr{R}(\mu) = \max\left\{\prod_{h=1}^{H}\prod_{r=1}^{n_h} Hn_h p_{h,r} : \sum_{r=1}^{n_h} p_{h,r} = \frac{1}{H}, \sum_{h=1}^{H}\sum_{r=1}^{n_h} p_{h,r}\left(Y_{[h],r} - \mu_h\right) = 0\right\}, \tag{9}$$

and the empirical log-likelihood ratio statistic is

$$\mathrm{EL}_{\mathrm{RSS}}(\mu) = -2\log \mathcal{R}(\mu) = 2\sum_{h=1}^{H}\sum_{r=1}^{n_h} \log\left(\frac{\lambda_h + v\left(Y_{[h],r} - \mu_h\right)}{Hn_h}\right), \tag{10}$$

where $p_{h,r}$, λ_h, and v are the solution to (6), (7) and (8).

The following theorem shows the asymptotic distribution of $\mathrm{EL}_{\mathrm{RSS}}(\mu)$ is a chi-square distribution with one degree of freedom. The detailed proof is presented in Appendix A.

Theorem 1. Assume that (i) $\mathbb{E}(Y) = \mu$, (ii) $\mathbb{E}|Y|^3 < \infty$, and (iii) $n_h/n \to q_h \in (0, 1)$ as $n \to \infty$, for every $h = 1, 2, \cdots, H$, where $n = \sum_{h=1}^{H} n_h$. For fixed H, as $n \to \infty$,

$$\mathrm{EL}_{\mathrm{RSS}}(\mu) \xrightarrow{d} \chi_1^2.$$

One can use Theorem 1 to test the hypothesis $H_0 : \mu = \mu_0$ with both balanced and unbalanced RSS data, where the testing statistic is $\mathrm{EL}_{\mathrm{RSS}}(\mu_0)$ and its asymptotic null distribution is the chi-square distribution with one degree of freedom.

3 Simulation

3.1 Existing Methods for Comparison

The problem of testing a population mean μ, $\mathcal{H}_0 : \mu = \mu_0$, is a fundamental question in statistics and various methods have been proposed for RSS in the literature. Below, we review the three most popular methods to test the population mean of RSS data.

One common practice is to use the asymptotic pivotal method by Chen et al. (2006), based on the pivotal statistic defined as

$$\text{Pivot} = \frac{\widehat{\mu}_{\text{RSS}} - \mu_0}{\widehat{\sigma}_{\widehat{\mu}_{\text{RSS}}}},$$

where

$$\widehat{\mu}_{\text{RSS}} = \frac{1}{H}\sum_{h=1}^{H}\frac{1}{n_h}\sum_{r=1}^{n_h} Y_{[h],r} \text{ and } \widehat{\sigma}^2_{\widehat{\mu}_{\text{RSS}}} = \frac{1}{H^2}\sum_{h=1}^{H}\frac{1}{n_h(n_h - 1)}\sum_{r=1}^{n_h}\left(Y_{[h],r} - \overline{Y}_{[h]}\right)^2$$

with $\overline{Y}_{[h]} = \sum_{r=1}^{n_h} Y_{[h],r}/n_h$. The two estimates above are an unbiased RSS estimate of the mean μ and a consistent estimate of the variance of $\widehat{\mu}_{\text{RSS}}$, respectively.

When the population distribution is symmetric, a test for the median M is equivalent to the test on the mean for such population. Therefore, one can use each RSS version of the sign test for BRSS (Bohn, 1998; Hettmansperger, 1995; Koti & Babu, 1996) and URSS (Barabesi, 2001). The statistic to test the null hypothesis $\mathcal{H}_0 : M = M_0$ is given by

$$S^+_{\text{RSS}} = \sum_{i=1}^{n} \mathrm{I}(y_i - M_0 > 0).$$

For BRSS, under the null hypothesis,

$$n^{-1/2}\left(S^+_{\text{RSS}} - \frac{n}{2}\right) \xrightarrow{d} N\left(0, \frac{\eta^2}{4}\right),$$

where

$$\eta^2 = 1 - \frac{4}{H}\sum_{h=1}^{H}\left\{B\left(h, H - h + 1, \frac{1}{2}\right) - \frac{1}{2}\right\}^2$$

and $B(h, s, q)$ is the cumulative distribution function of the beta distribution with parameters h and s for $0 \leq q \leq 1$. For URSS, under the null hypothesis,

$$\left(S^+_{\text{RSS}} - \sum_{h=1}^{H} n_h(1 - \beta_h)\right) \xrightarrow{d} N\left(0, \sum_{h=1}^{H} n_h\beta_h(1 - \beta_h)\right),$$

where

$$\beta_h = B\left(h, H - h + 1, \frac{1}{2}\right).$$

Baklizi (2009) proposed an EL ratio test (LRT) for the population mean and quantiles for BRSS (i.e., $n_h = m$). In the paper, he considered the empirical distribution having mass probability $p_{h,r}$ on $Y_{[h],r}$, for $h = 1, \ldots, H$ and $r = 1, \ldots, m$, and constructed the likelihood by treating RSS data as SRS data. The EL by Baklizi (2009), notated as $\mathrm{EL}_{\mathrm{RSS}}^{\mathrm{B}}$, maximises

$$\prod_{h=1}^{H}\prod_{r=1}^{m} p_{h,r} \tag{11}$$

under the constraints that the mean equals to μ and the sum of probabilities equals to 1 as

$$\sum_{h=1}^{H}\sum_{r=1}^{m} p_{h,r} = 1 \quad \text{and} \quad \sum_{h=1}^{H}\sum_{r=1}^{m} p_{h,r} Y_{]h]},\ r = \mu, \tag{12}$$

and achieves its maximum when

$$\prod_{h=1}^{H}\prod_{r=1}^{m} p_{h,r} \quad \text{with} \quad p_{h,r} = \frac{1}{mH}\frac{1}{1+\lambda\left(Y_{[h],r} - \mu\right)}$$

where λ, the Lagrange multiplier, is the solution of

$$\frac{1}{mH}\sum_{h=1}^{H}\sum_{r=1}^{m}\frac{Y_{[h],r} - \mu}{1+\lambda\left(Y_{[h],r} - \mu\right)} = 0.$$

In sequel, to test the hypothesis $\mathcal{H}_0 : \mu = \mu_0$, Baklizi (2009) suggested to use

$$\mathrm{EL}_{\mathrm{RSS}}^{\mathrm{B}}(\mu_0) = c \times 2\sum_{h=1}^{H}\sum_{r=1}^{m}\log\left[1+\lambda\left(Y_{[h],r} - \mu_0\right)\right] \tag{13}$$

with

$$c = \frac{\sum_{h=1}^{H}\sigma_{[h]}^2 + \sum_{h=1}^{H}\left(\mu_{[h]} - \mu_0\right)^2}{\sum_{h=1}^{H}\sigma_{[h]}^2},$$

and showed that (13) converges to χ_1^2 distribution under the null. In the above, $\mu_{[h]}$ and $\sigma_{[h]}^2$ are the mean and variance of the h-th order statistics, respectively. Here, we remark that (13) does not take into the ranking information in building the EL and is only applicable to BRSS.

3.2 Numerical Comparison

We numerically investigate the size and power of the proposed test $\mathrm{EL}_{\mathrm{RSS}}$ for the null hypothesis $\mathcal{H}_0 : \mu = \mu_0$, and compare it with the three existing methods reviewed in Section 3.1: the pivotal method (Pivot), the sign test assuming a symmetric underlying distribution ($\mathrm{S}_{\mathrm{RSS}}^{+}$), and the empirical LRT based on BRSS of Baklizi (2009) ($\mathrm{EL}_{\mathrm{RSS}}^{\mathrm{B}}$). Further, we refer to the empirical LRT based on SRS as $\mathrm{EL}_{\mathrm{SRS}}$.

The test $\mathrm{EL}_{\mathrm{SRS}}$ is a SRS counterpart to the proposed procedure by Owen (2001). Suppose we have n observations with $z_1 \le z_2 \le \cdots \le z_n$ by SRS. The empirical likelihood ratio statistic for this SRS is

$$\mathrm{EL}_{\mathrm{SRS}} = \prod_{i=1}^{n}\left\{1+\lambda(z_i - \mu_0)\right\}^{-1}$$

where λ is the solution to

$$\sum_{i=1}^{n} \frac{z_i - \mu_0}{1 + \lambda(z_i - \mu_0)} = 0,$$

and under the null hypothesis $\mathcal{H}_0 : \mu = \mu_0$, $-2\log \mathrm{EL}_{\mathrm{SRS}}$ is asymptotically chi-squared distributed with degrees of freedom 1.

We consider both BRSS and URSS with the set size $H \in \{2, 4\}$. The number of cycles is $n_h = m \in \{8, 16\}$ for BRSS and $n_h \in \{6, 10, 12, 20\}$ for URSS so that the total sample size is $n \in \{16, 32, 64\}$. For URSS, we do not consider Baklizi's method ($\mathrm{EL}^{\mathrm{B}}_{\mathrm{RSS}}$) because it is not valid for URSS. The RSS datasets were generated from four distributions: the normal distribution $N(0, 1)$, the t distribution t_5, the gamma distribution $GAM(3, 1)$ and the lognormal distribution $LN(0, 0.693)$. All distributions are shifted having mean δ. We use three mean values $\delta \in \{0, 0.25, 0.5\}$. The case $\delta = 0$ allows us to examine the unbiasedness of the test size and the others are to assess power.

On the other hand, we also investigate the performance of the tests by varying ranking quality. To simulate the imperfect ranking, we rank units through a latent variable Y^* based on a linear model $Y = Y^* + \epsilon$, where ϵ is the error term from a normal distribution with mean 0 and variance σ_ϵ^2 and independent with Y^*. Then, the correlation between Y and Y^*, $\rho = \sqrt{\sigma^2 - \sigma_\epsilon^2}/\sigma$, reflects the ranking accuracy, where σ^2 is the variance of Y. Here, the variance σ_ϵ^2 is set to make $\rho \in \{0.7, 0.9, 1\}$, where $\rho = 1$ means perfect ranking.

For each case, we generate $K = 5,000$ RSS data sets and compute statistics and p values for each method. The p values p_k, $k = 1, 2, \cdots, K$, are based on their asymptotic null distributions. We approximate the power at a given significance level α by $\sum_{k=1}^{K} I\{p_k \le \alpha\}/K$, where α is chosen to be 0.05. We also report size-corrected power in parenthesis in tables and compare it between the methods in the following. Note that, because the sign test utilises only the sign value instead of the absolute value, the statistics have only a finite number of discrete values. For this reason, the empirically corrected sizes are not at the significant level α in some cases.

The performance of the four methods depends on the underlying distribution. Table 1 reports the sizes and powers of tests under BRSS and URSS assuming perfect ranking for symmetric underlying distributions; normal and t distribution. From these tables, we find that Pivot is the best or second best and $\mathrm{S}^+_{\mathrm{RSS}}$ is the worst in both size and power in all cases. We recall Pivot assumes a symmetric underlying distribution. Next, when we compare two RSS-based empirical LRTs, $\mathrm{EL}_{\mathrm{RSS}}$ and $\mathrm{EL}^{\mathrm{B}}_{\mathrm{RSS}}$, the powers of $\mathrm{EL}^{\mathrm{B}}_{\mathrm{RSS}}$ tend to be higher than those of $\mathrm{EL}_{\mathrm{RSS}}$, but their differences are small. The results for the cases with imperfect ranking are reported in Tables B1–B4, which are consistent with those of perfect ranking.

Table 2 reports the sizes and powers of the tests for BRSS and URSS assuming perfect ranking when the data distribution is asymmetric, gamma and lognormal distribution. For these cases, we do not consider the sign test $\mathrm{S}^+_{\mathrm{RSS}}$ because median M and mean μ are not equal under asymmetric distributions. From these tables, we first observe that $\mathrm{EL}_{\mathrm{RSS}}$ outperforms Pivot in power regardless of RSS design (the allocations of samples). Second, the powers of two RSS-based EL methods, $\mathrm{EL}_{\mathrm{RSS}}$ and $\mathrm{EL}^{\mathrm{B}}_{\mathrm{RSS}}$, are similar to each other for BRSS. For a lognormal distribution, the former tends to exhibit higher power compared with the latter, yet for a gamma distribution, the latter exhibits higher power. Nonetheless, the differences between them are minimal. Third, the EL methods based on RSS show higher power than that based on SRS for most cases. Tables B5 and B6 report the results of imperfect ranking, which are again consistent with those of perfect ranking.

We have made three common observations regardless of the distribution used. First, the size of $\mathrm{EL}_{\mathrm{RSS}}$ is generally smaller than $\mathrm{EL}^{\mathrm{B}}_{\mathrm{RSS}}$, but the difference between them is negligible. They

Table 1. *With RSS under normal and t distribution assuming perfect ranking, approximated size and power of four RSS-based testing methods (Pivot,* S^{+}_{RSS}*,* EL^{B}_{RSS} *and* EL_{RSS}*) and SRS-based testing method (*EL_{SRS}*).*

			Normal Dist.					*t Dist.*				
H	n_h	δ	EL_{SRS}	*Pivot*	S^{+}_{RSS}	EL^{B}_{RSS}	EL_{RSS}	EL_{SRS}	*Pivot*	S^{+}_{RSS}	EL^{B}_{RSS}	EL_{RSS}
2	(8,8)	0	0.076 (0.050)	0.067 (0.050)	0.041 (0.143)	0.101 (0.050)	0.090 (0.050)	0.086 (0.050)	0.068 (0.050)	0.043 (0.147)	0.111 (0.050)	0.102 (0.050)
		0.25	0.211 (0.161)	0.245 (0.203)	0.126 (0.301)	0.298 (0.200)	0.284 (0.203)	0.187 (0.128)	0.194 (0.161)	0.121 (0.287)	0.260 (0.157)	0.248 (0.157)
		0.5	0.529 (0.456)	0.686 (0.627)	0.404 (0.647)	0.738 (0.618)	0.719 (0.619)	0.415 (0.327)	0.512 (0.456)	0.358 (0.600)	0.579 (0.434)	0.566 (0.432)
	(16,16)	0	0.068 (0.050)	0.065 (0.050)	0.064 (0.064)	0.077 (0.050)	0.069 (0.050)	0.063 (0.050)	0.057 (0.050)	0.062 (0.062)	0.077 (0.050)	0.070 (0.050)
		0.25	0.307 (0.254)	0.414 (0.366)	0.296 (0.296)	0.448 (0.356)	0.425 (0.361)	0.241 (0.216)	0.289 (0.271)	0.265 (0.265)	0.326 (0.248)	0.308 (0.252)
		0.5	0.799 (0.757)	0.930 (0.911)	0.764 (0.764)	0.938 (0.905)	0.934 (0.907)	0.613 (0.580)	0.760 (0.743)	0.722 (0.722)	0.780 (0.707)	0.757 (0.703)
4	(8,8,8,8)	0	0.061 (0.050)	0.058 (0.050)	0.034 (0.099)	0.070 (0.050)	0.074 (0.050)	0.064 (0.050)	0.058 (0.050)	0.036 (0.098)	0.077 (0.050)	0.086 (0.050)
		0.25	0.294 (0.272)	0.583 (0.555)	0.269 (0.437)	0.611 (0.551)	0.627 (0.553)	0.233 (0.198)	0.400 (0.378)	0.232 (0.397)	0.438 (0.364)	0.449 (0.360)
		0.5	0.807 (0.784)	0.991 (0.988)	0.786 (0.898)	0.991 (0.987)	0.992 (0.988)	0.616 (0.572)	0.884 (0.875)	0.736 (0.865)	0.888 (0.847)	0.894 (0.845)
	(16,16,16,16)	0	0.051 (0.050)	0.058 (0.050)	0.061 (0.061)	0.064 (0.050)	0.065 (0.050)	0.065 (0.050)	0.048 (0.050)	0.055 (0.055)	0.060 (0.050)	0.063 (0.050)
		0.25	0.514 (0.507)	0.862 (0.844)	0.609 (0.609)	0.869 (0.841)	0.872 (0.838)	0.359 (0.319)	0.635 (0.638)	0.573 (0.573)	0.642 (0.614)	0.641 (0.607)
		0.5	0.976 (0.974)	1.000 (1.000)	0.991 (0.991)	1.000 (1.000)	1.000 (1.000)	0.856 (0.829)	0.984 (0.984)	0.978 (0.978)	0.983 (0.980)	0.976 (0.972)
2	(6,10)	0	0.076 (0.050)	0.076 (0.050)	0.043 (0.146)	–	0.104 (0.050)	0.086 (0.050)	0.074 (0.050)	0.041 (0.147)	–	0.118 (0.050)
		0.25	0.211 (0.161)	0.255 (0.188)	0.112 (0.291)	–	0.296 (0.188)	0.187 (0.128)	0.208 (0.164)	0.113 (0.281)	–	0.270 (0.149)
		0.5	0.529 (0.456)	0.644 (0.555)	0.352 (0.607)	–	0.697 (0.543)	0.415 (0.327)	0.504 (0.445)	0.316 (0.572)	–	0.579 (0.409)
	(12,20)	0	0.068 (0.050)	0.060 (0.050)	0.059 (0.059)	–	0.067 (0.050)	0.063 (0.050)	0.057 (0.050)	0.062 (0.062)	–	0.074 (0.050)
		0.25	0.307 (0.254)	0.288 (0.262)	0.254 (0.254)	–	0.305 (0.260)	0.241 (0.216)	0.301 (0.286)	0.578 (0.578)	–	0.317 (0.268)
		0.5	0.799 (0.757)	0.909 (0.891)	0.725 (0.725)	–	0.915 (0.882)	0.613 (0.580)	0.731 (0.714)	0.692 (0.692)	–	0.728 (0.673)
	(10,6)	0	0.076 (0.050)	0.073 (0.050)	0.038 (0.133)	–	0.094 (0.050)	0.086 (0.050)	0.075 (0.050)	0.041 (0.137)	–	0.116 (0.050)
		0.25	0.211 (0.161)	0.240 (0.174)	0.139 (0.306)	–	0.290 (0.180)	0.187 (0.128)	0.174 (0.125)	0.137 (0.301)	–	0.234 (0.131)
		0.5	0.529 (0.456)	0.660 (0.572)	0.445 (0.666)	–	0.713 (0.567)	0.415 (0.327)	0.493 (0.405)	0.388 (0.617)	–	0.562 (0.395)
	(20,12)	0	0.068 (0.050)	0.062 (0.050)	0.065 (0.065)	–	0.072 (0.050)	0.063 (0.050)	0.057 (0.050)	0.062 (0.062)	–	0.076 (0.050)
		0.25	0.307 (0.254)	0.390 (0.348)	0.303 (0.303)	–	0.407 (0.352)	0.241 (0.216)	0.273 (0.249)	0.291 (0.291)	–	0.304 (0.235)
		0.5	0.799 (0.757)	0.912 (0.890)	0.782 (0.782)	—	0.919 (0.890)	0.613 (0.580)	0.749 (0.725)	0.743 (0.743)	-	0.755 (0.676)
4	(6,6,10,10)	0	0.061 (0.050)	0.065 (0.050)	0.055 (0.055)	–	0.090 (0.050)	0.064 (0.050)	0.063 (0.050)	0.052 (0.052)	–	0.095 (0.050)
		0.25	0.294 (0.272)	0.558 (0.508)	0.309 (0.309)	–	0.603 (0.503)	0.233 (0.198)	0.401 (0.363)	0.289 (0.289)	–	0.459 (0.340)
		0.5	0.807 (0.784)	0.982 (0.975)	0.820 (0.820)	–	0.986 (0.973)	0.616 (0.572)	0.857 (0.838)	0.775 (0.775)	–	0.878 (0.816)

(*Continues*)

Table 1 *(Continued)*

H	n_h	δ	*Normal Dist.*					*t Dist.*				
			EL_{SRS}	*Pivot*	S^+_{RSS}	EL^B_{RSS}	EL_{RSS}	EL_{SRS}	*Pivot*	S^+_{RSS}	EL^B_{RSS}	EL_{RSS}
	(12,12,20,20)	0	0.051 (0.050)	0.058 (0.050)	0.065 (0.065)	–	0.064 (0.050)	0.065 (0.050)	0.053 (0.050)	0.062 (0.062)	–	0.068 (0.050)
		0.25	0.514 (0.507)	0.841 (0.822)	0.586 (0.586)	–	0.849 (0.818)	0.359 (0.319)	0.614 (0.605)	0.541 (0.541)	–	0.613 (0.560)
		0.5	0.976 (0.974)	1.000 (1.000)	0.989 (0.989)	–	1.000 (1.000)	0.856 (0.829)	0.973 (0.971)	0.980 (0.980)	–	0.963 (0.953)
	(10,10,6,6)	0	0.061 (0.050)	0.056 (0.050)	0.048 (0.141)	–	0.081 (0.050)	0.064 (0.050)	0.067 (0.050)	0.055 (0.055)	–	0.099 (0.050)
		0.25	0.294 (0.272)	0.565 (0.543)	0.376 (0.566)	–	0.626 (0.541)	0.233 (0.198)	0.358 (0.309)	0.337 (0.337)	–	0.426 (0.303)
		0.5	0.807 (0.784)	0.988 (0.985)	0.884 (0.952)	–	0.992 (0.985)	0.616 (0.572)	0.878 (0.846)	0.836 (0.836)	–	0.895 (0.813)
	(20,20,12,12)	0	0.051 (0.050)	0.054 (0.050)	0.060 (0.060)	–	0.060 (0.050)	0.065 (0.050)	0.051 (0.050)	0.060 (0.060)	–	0.067 (0.050)
		0.25	0.514 (0.507)	0.841 (0.833)	0.622 (0.622)	–	0.851 (0.834)	0.359 (0.319)	0.615 (0.612)	0.595 (0.595)	–	0.636 (0.591)
		0.5	0.976 (0.974)	1.000 (1.000)	0.994 (0.994)	–	1.000 (1.000)	0.856 (0.829)	0.990 (0.990)	0.987 (0.987)	–	0.987 (0.982)

The size of tests is $\alpha = 0.05$. The parentheses are the empirically corrected powers for each method.

Table 2. *With RSS under gamma and lognormal distribution assuming perfect ranking, approximated size and power of four RSS-based testing methods (Pivot,* S^{+}_{RSS}, EL^{B}_{RSS} *and* EL_{RSS}*) and SRS-based testing method (*EL_{SRS}*).*

			Gamma Dist.				*Lognormal Dist.*			
H	n_h	δ	EL_{SRS}	*Pivot*	EL^{B}_{RSS}	EL_{RSS}	EL_{SRS}	*Pivot*	EL^{B}_{RSS}	EL_{RSS}
2	(8,8)	0	0.095 (0.050)	0.088 (0.050)	0.119 (0.050)	0.120 (0.050)	0.130 (0.050)	0.120 (0.050)	0.148 (0.050)	0.139 (0.050)
		0.25	0.117 (0.063)	0.099 (0.055)	0.166 (0.070)	0.160 (0.068)	0.259 (0.051)	0.176 (0.034)	0.372 (0.074)	0.351 (0.117)
		0.5	0.266 (0.165)	0.271 (0.166)	0.395 (0.226)	0.380 (0.215)	0.816 (0.467)	0.765 (0.357)	0.929 (0.642)	0.921 (0.719)
	(16,16)	0	0.067 (0.050)	0.062 (0.050)	0.075 (0.050)	0.069 (0.050)	0.101 (0.050)	0.088 (0.050)	0.106 (0.050)	0.094 (0.050)
		0.25	0.147 (0.107)	0.136 (0.108)	0.201 (0.140)	0.187 (0.150)	0.415 (0.131)	0.350 (0.181)	0.543 (0.335)	0.527 (0.387)
		0.5	0.437 (0.361)	0.489 (0.431)	0.604 (0.509)	0.586 (0.523)	0.979 (0.873)	0.983 (0.930)	0.997 (0.987)	0.997 (0.991)
4	(8,8,8,8)	0	0.071 (0.050)	0.073 (0.050)	0.084 (0.050)	0.090 (0.050)	0.092 (0.050)	0.093 (0.050)	0.108 (0.050)	0.113 (0.050)
		0.25	0.152 (0.101)	0.207 (0.150)	0.274 (0.199)	0.286 (0.196)	0.414 (0.180)	0.514 (0.302)	0.691 (0.484)	0.706 (0.495)
		0.5	0.450 (0.359)	0.693 (0.611)	0.778 (0.695)	0.790 (0.687)	0.979 (0.915)	0.998 (0.981)	1.000 (0.999)	1.000 (0.999)
	(16,16,16,16)	0	0.053 (0.050)	0.058 (0.050)	0.062 (0.050)	0.061 (0.050)	0.776 (0.050)	0.069 (0.050)	0.071 (0.050)	0.072 (0.050)
		0.25	0.233 (0.290)	0.381 (0.356)	0.441 (0.403)	0.449 (0.412)	0.675 (0.551)	0.860 (0.790)	0.927 (0.889)	0.932 (0.903)
		0.5	0.708 (0.702)	0.948 (0.940)	0.962 (0.954)	0.963 (0.955)	1.000 (0.999)	1.000 (1.000)	1.000 (1.000)	1.000 (1.000)
2	(6,10)	0	0.095 (0.050)	0.081 (0.050)	–	0.106 (0.050)	0.130 (0.050)	0.092 (0.050)	–	0.110 (0.050)
		0.25	0.117 (0.063)	0.102 (0.055)	–	0.159 (0.071)	0.259 (0.051)	0.187 (0.068)	–	0.359 (0.181)
		0.5	0.266 (0.165)	0.272 (0.176)	–	0.381 (0.219)	0.816 (0.467)	0.819 (0.567)	–	0.946 (0.852)
	(12,20)	0	0.067 (0.050)	0.064 (0.050)	–	0.069 (0.050)	0.101 (0.050)	0.078 (0.050)	–	0.076 (0.050)
		0.25	0.147 (0.107)	0.151 (0.120)	–	0.203 (0.165)	0.415 (0.131)	0.403 (0.267)	–	0.579 (0.462)
		0.5	0.437 (0.361)	0.514 (0.463)	–	0.605 (0.546)	0.979 (0.873)	0.992 (0.970)	–	0.999 (0.997)
	(10,6)	0	0.095 (0.050)	0.094 (0.050)	–	0.140 (0.050)	0.130 (0.050)	0.124 (0.050)	–	0.153 (0.050)
		0.25	0.117 (0.063)	0.099 (0.048)	–	0.169 (0.066)	0.259 (0.051)	0.144 (0.029)	–	0.313 (0.086)
		0.5	0.266 (0.165)	0.242 (0.129)	–	0.370 (0.168)	0.816 (0.467)	0.668 (0.258)	–	0.894 (0.550)
	(20,12)	0	0.067 (0.050)	0.077 (0.050)	–	0.082 (0.050)	0.101 (0.050)	0.102 (0.050)	–	0.109 (0.050)
		0.25	0.147 (0.107)	0.120 (0.077)	–	0.174 (0.108)	0.415 (0.131)	0.271 (0.088)	–	0.472 (0.271)
		0.5	0.437 (0.361)	0.435 (0.336)	–	0.525 (0.422)	0.979 (0.873)	0.953 (0.777)	–	0.993 (0.970)
4	(6,6,10,10)	0	0.071 (0.050)	0.062 (0.050)	–	0.079 (0.050)	0.092 (0.050)	0.072 (0.050)	–	0.093 (0.050)
		0.25	0.152 (0.101)	0.227 (0.194)	–	0.304 (0.239)	0.414 (0.180)	0.593 (0.471)	–	0.754 (0.640)
		0.5	0.450 (0.359)	0.734 (0.699)	–	0.815 (0.758)	0.979 (0.915)	0.999 (0.997)	–	1.000 (1.000)
	(12,12,20,20)	0	0.053 (0.050)	0.058 (0.050)	–	0.065 (0.050)	0.776 (0.050)	0.071 (0.050)	–	0.075 (0.050)
		0.25	0.233 (0.290)	0.432 (0.399)	–	0.498 (0.449)	0.675 (0.551)	0.915 (0.856)	–	0.958 (0.937)
		0.5	0.708 (0.702)	0.963 (0.955)	–	0.976 (0.968)	1.000 (0.999)	1.000 (1.000)	–	1.000 (1.000)

(*Continues*)

Table 2 *(Continued)*

H	n_h	δ	*Gamma Dist.* EL_{SRS}	*Gamma Dist.* *Pivot*	*Gamma Dist.* EL^{B}_{RSS}	*Gamma Dist.* EL_{RSS}	*Lognormal Dist.* EL_{SRS}	*Lognormal Dist.* *Pivot*	*Lognormal Dist.* EL^{B}_{RSS}	*Lognormal Dist.* EL_{RSS}
	(10,10,6,6)	0	0.071 (0.050)	0.081 (0.050)	–	0.110 (0.050)	0.092 (0.050)	0.100 (0.050)	–	0.130 (0.050)
		0.25	0.152 (0.101)	0.172 (0.111)	–	0.268 (0.148)	0.414 (0.180)	0.413 (0.177)	–	0.632 (0.372)
		0.5	0.450 (0.359)	0.616 (0.489)	–	0.731 (0.550)	0.979 (0.915)	0.991 (0.918)	–	1.000 (0.997)
	(20,20,12,12)	0	0.053 (0.050)	0.066 (0.050)	–	0.077 (0.050)	0.776 (0.050)	0.091 (0.050)	–	0.098 (0.050)
		0.25	0.233 (0.290)	0.319 (0.262)	–	0.398 (0.319)	0.675 (0.551)	0.757 (0.577)	–	0.875 (0.785)
		0.5	0.708 (0.702)	0.906 (0.864)	–	0.937 (0.904)	1.000 (0.999)	1.000 (1.000)	–	1.000 (1.000)

The size of tests is $\alpha = 0.05$. The parentheses are the empirically corrected powers for each method.

both tend to be greater than the nominal level of $\alpha = 0.05$ when the sample size is small, but they become closer to 0.05 as the sample size increases. Second, the proposed EL_{RSS} consistently demonstrates greater power compared with EL_{SRS} regardless of the sample allocation and ranking quality. Finally, we have found that the power of the RSS-based methods is asymmetrical in terms of sample allocation. For instance, the case with allocation (12, 20) has a higher power than the case with allocation (20, 12). This is because, in the study, the alternative hypothesis samples have a positive mean of $\delta > 0$. Thus, by allocating more samples to the second stratum [e.g. (12, 20)], we have more samples that are further from 0, which leads to a higher power in testing for $\delta = 0$ than the other allocation [e.g. (20, 12)]. This phenomenon has been observed for all cases under the gamma and lognormal distributions and for some cases with small sample sizes under normal and t distributions. The difference under the lognormal distribution is larger than under the gamma distribution as the skewness of the lognormal distribution with parameters 0 and 0.693 is greater than the skewness of the gamma distribution with parameters 3 and 1. It is widely recognised that well-designed URSS is more efficient than BRSS for skewed underlying distributions.

4 Examples and Applications

4.1 Bodyfat Data

We compare the mean testing methods using the body fat data of 252 men determined through underwater weighing and various body size measurements. The data were previously analysed in Wang et al. (2008). Our focus is on the mean percentage of body fat for the 252 men. To assess the influence of ranking errors, we perform ranking based on the percentage of body fat for perfect ranking, and on two other variables, abdominal circumference and chest circumference, for imperfect ranking. The two other variables, abdominal circumference and chest circumference, have correlations of $\rho = 0.81$ and $\rho = 0.7$ with the percentage of body fat, respectively.

The variable of interest Y is the percentage of bodyfat which has a mean of 19.151 with a standard deviation of 8.369. Figure 1 plots the histogram of the bodyfat variable and shows it is lightly skewed to the right. We add $\delta \in \{0, \pm 2, \pm 4, \pm 6\}$ to the data Y and generate 5 000 RSS data sets with $H = 2$ and the number of cycles $n_h = m \in \{8, 16\}$ for BRSS and

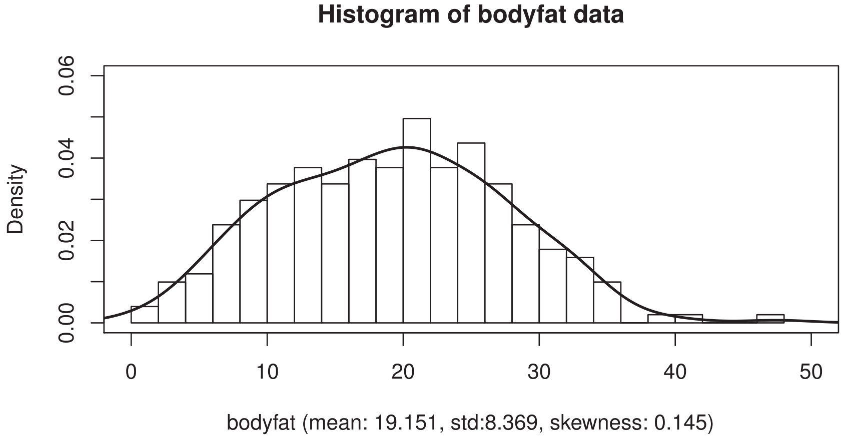

Figure 1. *Histogram of bodyfat data.*

$n_h \in \{6, 10, 12, 20\}$ for URSS so that $n \in \{16, 32\}$. We test hypotheses $H_0 : \mu = 19$ versus $H_1 : \mu \neq 19$ at a significant level $\alpha = 0.05$. The approximate power of the five methods is compared where EL^B_{RSS} is not considered for URSS.

Figure 2 (respectively, Figures C1 and C2) presents approximate power curves under perfect ranking with $\rho = 1$ (respectively, under imperfect ranking with $\rho = 0.81$ and $\rho = 0.70$). The results indicate that the proposed method EL_{RSS} outperforms the other methods. In addition, S^+_{RSS} shows lower power than SRS-based EL_{SRS} although it is based on RSS data. This is because the underlying distribution is asymmetric as we understand from the numerical study. All the

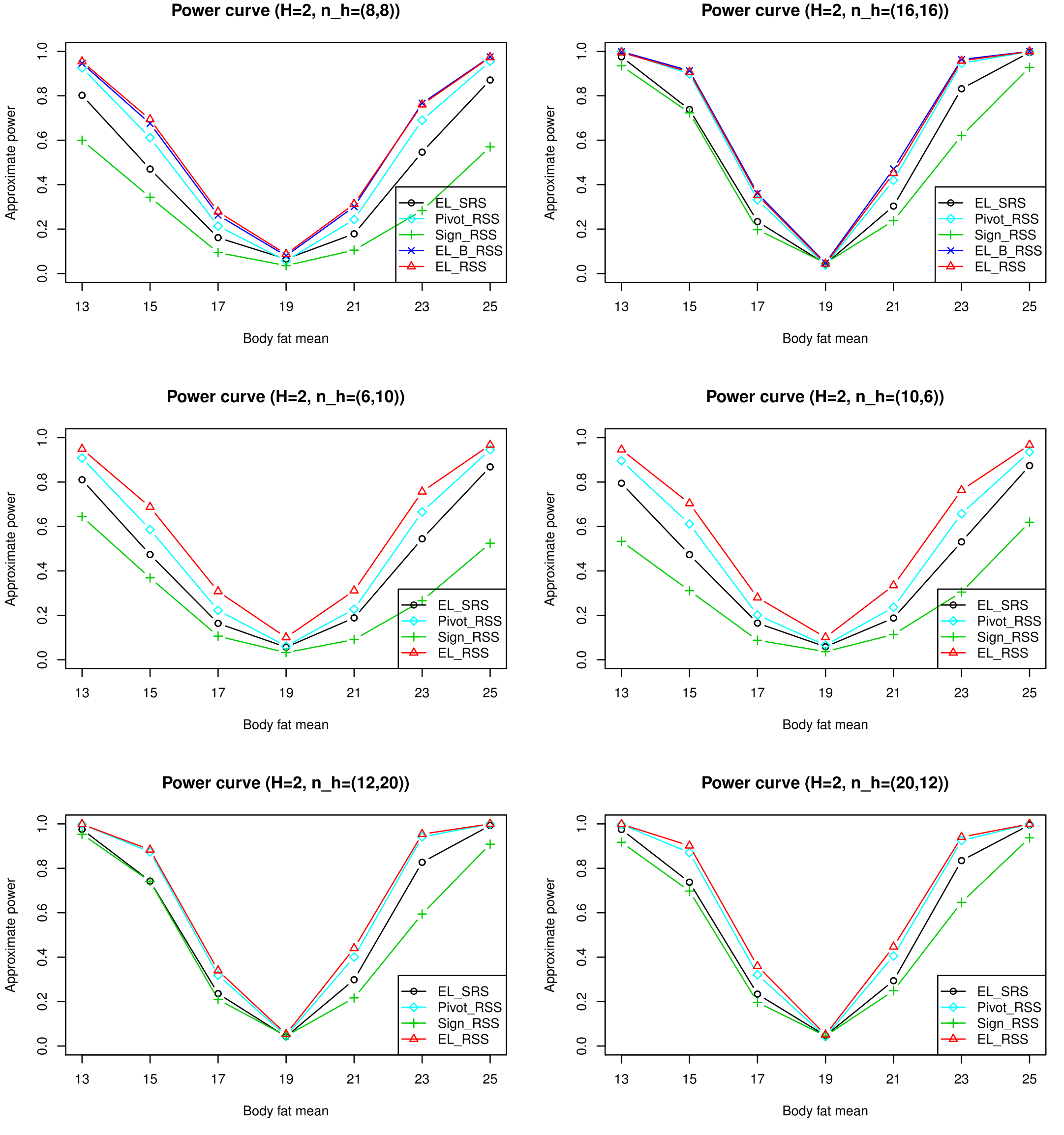

Figure 2. *Approximate power comparison of the four ranked set sampling (RSS)-based testing methods (Pivot, S^+_{RSS}, EL^B_{RSS} and EL_{RSS}) and the SRS-based testing method (EL_{SRS}) under perfect ranking with $H = 2$.*

findings above do not depend on the ranking accuracy and are consistent with simulation results in Section 4.

4.2 Tooth Size Data

We conduct a study on the tooth size of 295 individuals, focusing on four variables: the front teeth of the maxilla and mandible and the molars of the maxilla and mandible (Kim et al., 2005; Lee et al., 2007; Ozturk et al., 2023). We aim to assess the symmetry of tooth size. Our null hypothesis, $H_0: \mu^{ij}_{left} - \mu^{ij}_{right} = 0$, states that the size of left and right teeth are equal at a significance level of $\alpha = 0.05$. Here, i refers to the maxilla or mandible and j refers to the front or molar teeth. We use a paired t test to analyse the data, and the results are presented in Table 3. The sizes of all teeth except for the mandibular front teeth are found to be symmetrical at a significance level of $\alpha = 0.05$. Additionally, the histograms of the four paired differences in Figure 3 show that the three types of teeth are symmetrical, with means close to 0, indicated by the red dotted lines.

We generate 5 000 RSS data sets with $H = 2$, the number of cycles $n_h = m \in \{8, 16\}$ for BRSS and $n_h \in \{6, 10, 12, 20\}$ for URSS so that $n \in \{16, 32\}$, and consider perfect and imperfect ranking with $\rho \in \{1, 0.9, 0.7\}$ using linear ranking error model as in Section 3.2. We report the percentage (%) of rejected cases among 5 000 replicates for each procedure in Table 4. For URSS, $\mathrm{EL}^{\mathrm{B}}_{\mathrm{RSS}}$ is not considered for the comparison.

The results in Table 4 suggest that for all five methods, the null hypothesis of symmetrical tooth size is generally not rejected, except for the mandibular front teeth. Conversely, all five methods frequently reject the hypothesis of symmetrical mandibular front teeth. These findings align with those in Table 3. When comparing the testing methods, the $\mathrm{EL}_{\mathrm{RSS}}$ method appears to be more powerful when the ranking is perfect. However, as the quality of the ranking decreases, the differences between $\mathrm{EL}_{\mathrm{RSS}}$ and the other methods become smaller.

5 Extension to JEL

It is known that the EL approaches with nonlinear statistics have difficulty in computation. To overcome this problem, the jackknife EL (JEL) method, proposed under SRS by Jing et al. (2009), defines jackknife pseudo-values and treats them as independent and identically distributed (i.i.d.) samples. By simply applying the EL approach to their sample mean, it linearises the nonlinear statistics and facilitates the computations. Thus, likewise EL method, JEL is a distributional-free method and only requires that the pseudo values are i.i.d. samples. For this simplicity, there are numerous applications (An & Zhao, 2018; Feng & Peng, 2012; Sang

Table 3. *Description of Tooth size data. Mean and std denote the mean and standard deviation of the difference between left and right tooth size from 295 people.*

Data		*Difference* *Mean (std)*	*t*	*p value*
Maxilar	Front	0.002 (0.219)	0.167	0.867
	Molar	−0.022 (0.281)	−1.358	0.175
Mandibular	Front	0.060 (0.414)	2.473	0.014
	Molar	0.014 (0.267)	0.900	0.369

t and p values denote the t statistics and p value from the paired t test.

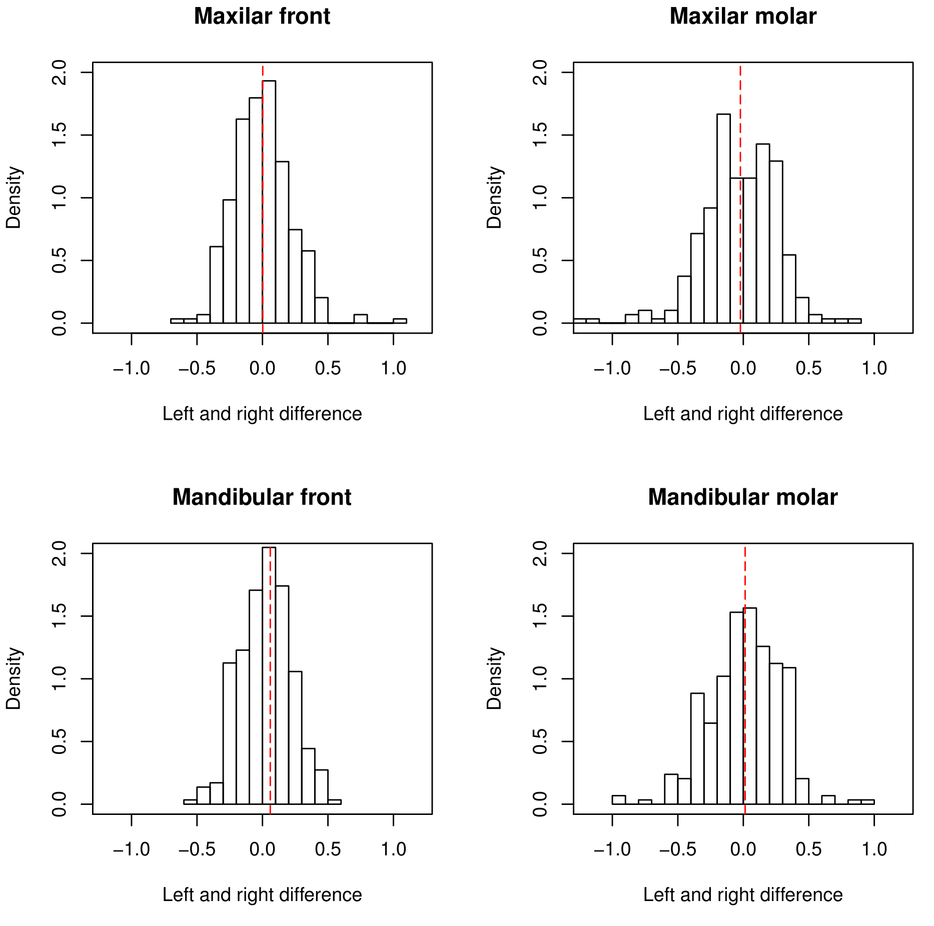

Figure 3. *Histograms of differences of left and right tooth size data. Red vertical lines denote the mean for each data.*

et al., 2021; Wang et al., 2021; Zhao et al., 2020). Recently, Zhang et al. (2016) propose the JEL approach with the RSS data, RSS-JEL, for the inference of the population mean and the mean difference between two groups.

Zhang et al. (2016) set

$$T_n = \frac{1}{H}\sum_{h=1}^{H}\overline{Y}_{[h]}$$

and, for each $i = 1, 2, \cdots, n$, uniquely express i as the sum $\sum_{h=0}^{h_i-1} n_h + j_i$ where $h_i \in \{1, 2, \cdots, H\}, j_i \in \{1, 2, \cdots, n_{h_i}\}$ and $n_0 = 0$. Then, let the leave-one-out statistics as

$$T_{n-1}^{(-i)} = \frac{1}{k}\sum_{h=1,\, h\neq h_i}^{H}\overline{Y}_{[h]} + \frac{1}{H(n_{h_i}-1)}\sum_{r=1,\, r\neq j_i}^{n_{h_i}} Y_{[h_i],\, r},\ i = 1, 2, \cdots, n$$

and define the jackknife pseudo-values by $V_i = nT_n - (n-1)T_{n-1}^{(-i)}$. By considering the values as i.i.d. samples as in Jing et al. (2009) with SRS data, Zhang et al. (2016) propose the jackknife empirical likelihood ratio statistic for this (SRS-like) data by

Table 4. *The percentage (%) of rejected cases by the four ranked set sampling-based testing methods (Pivot, S_{RSS}^{+}, EL_{RSS}^{B} and EL_{RSS}) and the SRS-based testing method (EL_{SRS}) under linear ranking error model with $H = 2$.*

				$\rho = 1$				$\rho = 0.9$				$\rho = 0.7$			
Data		n_h	EL_{SRS}	*Pivot*	S_{RSS}^{+}	EL_{RSS}^{B}	EL_{RSS}	*Pivot*	S_{RSS}^{+}	EL_{RSS}^{B}	EL_{RSS}	*Pivot*	S_{RSS}^{+}	EL_{RSS}^{B}	EL_{RSS}
Maxilar	Front	(8,8)	7.14	5.50	3.42	8.12	7.66	6.88	4.32	9.14	8.36	6.56	4.68	9.46	8.42
		(16,16)	4.98	3.72	5.04	5.18	4.66	4.90	6.34	5.74	5.30	4.84	7.30	5.50	4.70
		(6,10)	7.34	6.06	3.24	–	8.54	6.52	4.16	–	9.26	6.96	6.74	–	9.52
		(12,20)	4.98	4.32	5.32	–	5.44	4.90	7.22	–	5.20	4.84	11.50	–	4.90
		(10,6)	7.14	6.20	2.96	–	8.62	6.92	4.04	–	8.80	7.30	5.26	–	8.84
		(20,12)	4.98	4.60	4.64	–	5.34	4.90	5.76	–	5.78	4.84	7.40	–	5.12
	Molar	(8,8)	8.36	6.66	4.00	10.96	10.28	7.60	4.46	10.76	9.88	8.12	5.68	11.26	9.90
		(16,16)	7.78	6.04	5.18	9.00	8.08	6.88	5.74	8.72	7.88	6.78	8.50	9.18	7.40
		(6,10)	7.98	7.22	3.70	–	11.74	7.76	4.92	–	10.26	8.00	7.72	–	10.50
		(12,20)	7.78	5.68	5.80	–	7.82	6.88	6.30	–	7.68	6.78	11.98	–	7.40
		(10,6)	8.24	7.30	3.26	–	10.94	8.46	3.76	–	10.32	7.92	5.64	–	10.24
		(20,12)	7.78	6.40	5.14	–	8.52	6.88	4.62	–	8.72	6.78	7.86	–	8.12
Mandibular	Front	(8,8)	16.24	12.02	6.98	22.12	21.42	16.32	8.92	20.66	18.72	15.32	9.86	20.38	17.98
		(16,16)	21.14	13.84	14.16	28.86	28.62	22.50	15.50	26.78	25.56	22.46	17.98	27.10	25.48
		(6,10)	15.96	10.62	5.32	–	21.86	15.80	6.42	–	20.32	16.84	6.20	–	18.68
		(12,20)	21.14	12.90	13.50	–	29.10	22.50	10.54	–	26.58	22.46	10.16	–	25.30
		(10,6)	15.90	12.00	6.72	–	20.26	15.86	12.28	–	19.74	15.88	16.54	–	19.02
		(20,12)	21.14	14.58	14.96	–	26.78	22.50	23.22	–	22.58	22.46	30.64	–	22.72
	Molar	(8,8)	7.60	6.72	4.34	9.48	9.02	8.26	5.02	10.16	9.16	7.24	6.16	10.64	9.44
		(16,16)	6.88	5.84	8.04	7.08	6.62	6.28	9.44	7.12	6.56	6.10	10.36	7.72	6.86
		(6,10)	8.50	8.20	4.50	–	11.14	8.22	4.54	–	9.94	8.16	5.58	–	11.30
		(12,20)	6.88	6.62	7.60	–	7.56	6.28	7.30	–	6.94	6.10	7.40	–	7.50
		(10,6)	7.32	7.42	4.42	–	9.92	7.34	7.12	–	10.06	7.90	8.36	–	9.82
		(20,12)	6.88	5.54	7.78	–	6.44	6.28	11.84	–	6.16	6.10	14.26	–	6.70

$$\mathrm{JEL}_{\mathrm{RSS}}^{\mathrm{Z}} = \prod_{i=1}^{n} \{1 + \lambda(V_i - \mu_0)\}^{-1}$$

where λ is the solution to

$$\sum_{i=1}^{n} \frac{V_i - \mu_0}{1 + \lambda(V_i - \mu_0)} = 0,$$

and show that $-2\log\mathrm{JEL}_{\mathrm{RSS}}^{\mathrm{Z}}$ converges in distribution to χ_1^2 under the null hypothesis $\mathcal{H}_0 : \mu = \mu_0$.

However, in RSS, it is easy to see that the distribution of the value V_i depends on the stratum h_i and so V_is are not identically distributed. Thus, we newly define the jackknife pseudo-values for each stratum and extend the new EL method for the population mean in Section 2 to JEL method for RSS data.

5.1 New JEL Method for RSS

Let $T_{h,n_h} = T\left(X_{[h]1}, \cdots, X_{[h]n_h}\right)$ be a consistent estimator of the parameter θ_h. For the h-th strata, define the jackknife pseudo-values for the h-th strata as

$$V_{h,r} = n_h T_{h,n_h} - (n_h - 1) T_{h,n_h-1}^{(-r)} \tag{14}$$

where $T_{h,n_h-1}^{(-r)} = T\left(Y_{[h],1}, \cdots, Y_{[h],r-1}, Y_{[h],r+1}, \cdots, Y_{[h],n_h}\right)$ is the statistic computed on the sample of $n_h - 1$ formed from the original data set by deleting the i-th data value in the h-th strata ($h = 1, 2, \cdots, H$). The jackknife estimator of θ_h is simply the average of the pseudo-values

$$T_{h,n_h}^{Jack} = \frac{1}{n_h} \sum_{r=1}^{n_h} V_{h,r}.$$

We can easily show that

$$T_{h,n_h} = T_{h,n_h}^{Jack} \tag{15}$$

is a sample average of approximately independent random variable $V_{h,r}$ s.

Let $p_h = \left\{p_{h,1}, \cdots, p_{h,n_h}\right\}$ be the probabilities such that $\sum_{r=1}^{n_h} p_{h,r} = \frac{1}{H}$. Then over $h = 1, \cdots, H$, the JEL with RSS for θ is given by

$$\mathscr{L}(\theta) = \max\left\{ \prod_{h=1}^{H} \prod_{r=1}^{n_h} H n_h p_{h,r} : \sum_{r=1}^{n_h} p_{h,r} = \frac{1}{H}, \sum_{h=1}^{H} \sum_{r=1}^{n_h} p_{h,r}\left(V_{h,r} - \theta_h\right) = 0 \right\}.$$

Then, the jackknife empirical log-likelihood ratio statistic is

$$\mathrm{JEL}_{\mathrm{RSS}}(\theta) = -2\log \mathcal{L}(\theta) = 2 \sum_{h=1}^{H} \sum_{r=1}^{n_h} \log\left(\frac{\lambda_h + v\left(V_{h,r} - \theta_h\right)}{H n_h} \right)$$

where

$$p_{h,r} = \frac{1}{\lambda_h + v\left(V_{h,r} - \theta_h\right)}$$

and the Lagrange multipliers, λ_h and v, are the solutions of

Table 5. *Approximated size and power of two RSS JEL methods* (JEL^{Z}_{RSS} *and* EL_{RSS}).

			Normal Dist.		*t Dist.*		*Gamma Dist.*		*Lognormal Dist.*	
H	n_h	δ	JEL^{Z}_{RSS}	EL_{RSS}	JEL^{Z}_{RSS}	EL_{RSS}	JEL^{Z}_{RSS}	EL_{RSS}	JEL^{Z}_{RSS}	EL_{RSS}
2	(8,8)	0	0.062(0.050)	0.084(0.050)	0.075(0.050)	0.102(0.050)	0.082(0.050)	0.114(0.050)	0.111(0.050)	0.139(0.050)
		0.25	0.242(0.212)	0.286(0.212)	0.201(0.151)	0.242(0.152)	0.120(0.066)	0.165(0.062)	0.244(0.057)	0.334(0.115)
		0.5	0.664(0.622)	0.715(0.623)	0.511(0.444)	0.576(0.441)	0.291(0.186)	0.367(0.176)	0.881(0.545)	0.928(0.720)
	(16,16)	0	0.057(0.050)	0.066(0.050)	0.060(0.050)	0.070(0.050)	0.065(0.050)	0.073(0.050)	0.094(0.050)	0.096(0.050)
		0.25	0.411(0.384)	0.436(0.385)	0.290(0.257)	0.308(0.251)	0.170(0.133)	0.197(0.145)	0.476(0.211)	0.529(0.380)
		0.5	0.918(0.904)	0.926(0.904)	0.734(0.704)	0.743(0.688)	0.548(0.491)	0.584(0.508)	0.994(0.970)	0.996(0.990)
4	(8,8,8,8)	0	0.050(0.050)	0.083(0.050)	0.056(0.050)	0.087(0.050)	0.054(0.050)	0.083(0.050)	0.080(0.050)	0.113(0.050)
		0.25	0.548(0.547)	0.633(0.546)	0.374(0.357)	0.448(0.345)	0.203(0.189)	0.289(0.201)	0.559(0.328)	0.708(0.500)
		0.5	0.983(0.983)	0.992(0.982)	0.864(0.852)	0.894(0.835)	0.697(0.676)	0.792(0.695)	1.000(0.997)	1.000(1.000)
	(16,16,16,16)	0	0.046(0.050)	0.059(0.050)	0.054(0.050)	0.069(0.050)	0.051(0.050)	0.067(0.050)	0.067(0.050)	0.075(0.050)
		0.25	0.841(0.853)	0.866(0.853)	0.611(0.604)	0.637(0.588)	0.414(0.408)	0.463(0.421)	0.907(0.853)	0.932(0.903)
		0.5	1.000(1.000)	1.000(1.000)	0.985(0.985)	0.978(0.971)	0.951(0.950)	0.962(0.952)	1.000(1.000)	1.000(1.000)
2	(6,10)	0	0.071(0.050)	0.098(0.050)	0.078(0.050)	0.118(0.050)	0.070(0.050)	0.100(0.050)	0.098(0.050)	0.116(0.050)
		0.25	0.240(0.196)	0.304(0.188)	0.205(0.157)	0.255(0.144)	0.122(0.089)	0.158(0.075)	0.296(0.119)	0.363(0.155)
		0.5	0.619(0.553)	0.699(0.554)	0.499(0.417)	0.579(0.410)	0.319(0.253)	0.387(0.219)	0.917(0.779)	0.943(0.817)
	(12,20)	0	0.054(0.050)	0.068(0.050)	0.065(0.050)	0.082(0.050)	0.063(0.050)	0.072(0.050)	0.081(0.050)	0.080(0.050)
		0.25	0.371(0.354)	0.407(0.354)	0.295(0.259)	0.315(0.251)	0.175(0.149)	0.200(0.155)	0.549(0.384)	0.580(0.487)
		0.5	0.890(0.880)	0.912(0.882)	0.706(0.679)	0.715(0.654)	0.576(0.535)	0.603(0.547)	0.997(0.993)	0.998(0.996)
	(10,6)	0	0.064(0.050)	0.096(0.050)	0.076(0.050)	0.116(0.050)	0.083(0.050)	0.146(0.054)	0.122(0.050)	0.170(0.050)
		0.25	0.223(0.186)	0.283(0.189)	0.170(0.129)	0.234(0.136)	0.104(0.060)	0.173(0.077)	0.165(0.024)	0.321(0.086)
		0.5	0.626(0.574)	0.713(0.586)	0.474(0.401)	0.558(0.401)	0.226(0.130)	0.353(0.159)	0.750(0.213)	0.888(0.528)
	(20,12)	0	0.056(0.050)	0.070(0.050)	0.067(0.050)	0.079(0.050)	0.069(0.050)	0.085(0.050)	0.100(0.050)	0.103(0.050)
		0.25	0.367(0.340)	0.406(0.344)	0.255(0.217)	0.290(0.217)	0.142(0.091)	0.182(0.123)	0.358(0.006)	0.465(0.294)
		0.5	0.902(0.886)	0.919(0.889)	0.721(0.678)	0.746(0.665)	0.435(0.330)	0.513(0.402)	0.988(0.193)	0.994(0.977)
4	(6,6,10,10)	0	0.049(0.050)	0.085(0.050)	0.056(0.050)	0.095(0.050)	0.051(0.050)	0.076(0.050)	0.070(0.050)	0.097(0.050)
		0.25	0.505(0.511)	0.601(0.519)	0.369(0.361)	0.451(0.346)	0.226(0.226)	0.305(0.234)	0.658(0.535)	0.754(0.611)
		0.5	0.974(0.974)	0.987(0.978)	0.832(0.828)	0.876(0.809)	0.735(0.734)	0.817(0.744)	1.000(1.000)	1.000(1.000)
	(12,12,20,20)	0	0.048(0.050)	0.065(0.050)	0.053(0.050)	0.069(0.050)	0.045(0.050)	0.058(0.050)	0.066(0.050)	0.072(0.050)
		0.25	0.822(0.829)	0.855(0.830)	0.582(0.575)	0.607(0.550)	0.448(0.466)	0.492(0.469)	0.942(0.918)	0.953(0.931)
		0.5	1.000(1.000)	1.000(1.000)	0.974(0.972)	0.960(0.950)	0.965(0.969)	0.974(0.969)	1.000(1.000)	1.000(1.000)
	(10,10,6,6)	0	0.057(0.050)	0.095(0.050)	0.055(0.050)	0.100(0.050)	0.061(0.050)	0.108(0.050)	0.089(0.050)	0.133(0.050)
		0.25	0.510(0.481)	0.621(0.490)	0.332(0.320)	0.439(0.324)	0.157(0.131)	0.281(0.159)	0.391(0.069)	0.641(0.363)
		0.5	0.978(0.973)	0.992(0.975)	0.854(0.846)	0.896(0.824)	0.584(0.532)	0.742(0.571)	0.997(0.720)	1.000(0.997)
	(20,20,12,12)	0	0.050(0.050)	0.066(0.050)	0.048(0.050)	0.068(0.050)	0.052(0.050)	0.072(0.050)	0.078(0.050)	0.096(0.050)
		0.25	0.825(0.826)	0.857(0.829)	0.579(0.591)	0.627(0.581)	0.325(0.318)	0.400(0.334)	0.798(0.556)	0.870(0.790)
		0.5	1.000(1.000)	1.000(1.000)	0.988(0.988)	0.982(0.977)	0.917(0.914)	0.947(0.920)	1.000(1.000)	1.000(1.000)

The size of tests is $\alpha = 0.05$. The parentheses are the empirically corrected powers for each method.

$$\sum_{r=1}^{n_h} \frac{1}{\lambda_h + v\left(V_{h,r} - \theta_h\right)} = \frac{1}{H}$$

and

$$\sum_{h=1}^{H} \sum_{r=1}^{n_h} \frac{V_{h,r} - \theta_h}{\lambda_h + v\left(V_{h,r} - \theta_h\right)} = 0.$$

Moreover, if $E\left(\sum_{h=1}^{H} V_{h,r}\right) = \theta$, $E\left|V_{h,r}\right|^3 \leq \infty$ and $n_h/n \to q_h$ as $n \to \infty$ for every $h = 1, 2, \cdots, H$, where $n = \sum_{h=1}^{H} n_h$, we have

$$\mathrm{JEL}_{\mathrm{RSS}}(\theta) \xrightarrow{\mathrm{d}} \chi_1^2$$

as $n \to \infty$ for fixed H as in Theorem 1 in Section 2.

For the population mean inference, let $\theta = \mu$ and $Y_{[h],1}, Y_{[h],2}, \cdots, Y_{[h],n_h} \sim F_h(\cdot)$ with $\theta_h = E\left(Y_{[h],r}\right)$. Then

$$T_{h,n_h} = \frac{1}{n_h} \sum_{1 \leq r \leq n_h} Y_{[h],r}$$

and

$$T_{h,n_h - 1}^{(-r)} = \frac{1}{n_h - 1} \sum_{1 \leq i \leq n_h, i \neq r} Y_{[h],i}$$

define the jackknife pseudo-values in (14), and we can show the property (15). By simple calculations, in the one-sample mean testing, we can show that $\mathrm{JEL}_{\mathrm{RSS}}(\mu) = \mathrm{EL}_{\mathrm{RSS}}(\mu)$.

5.2 Simulation Studies

We numerically investigate the size and power of the proposed test $\mathrm{JEL}_{\mathrm{RSS}}$ (equals to $\mathrm{EL}_{\mathrm{RSS}}$) for the null hypothesis $\mathcal{H}_0 : \mu = \mu_0$ and compare it with the test $\mathrm{JEL}_{\mathrm{RSS}}^{Z}$ by Zhang et al. (2016) for all cases considered in Section 3.2.

Table 5 reports the sizes and powers of the tests for BRSS and URSS when the underlying distribution is normal, *t* distribution, gamma and lognormal distributions; the normal and *t* distributions are symmetric and the gamma and lognormal distributions are asymmetric. For the symmetric distributions, we find that $\mathrm{EL}_{\mathrm{RSS}}$ exhibits higher power for normal distribution but smaller power for *t* distribution. However, the differences between the two are small. For the asymmetric distributions, $\mathrm{JEL}_{\mathrm{RSS}}$ (equals to $\mathrm{EL}_{\mathrm{RSS}}$) always outperforms $\mathrm{JEL}_{\mathrm{RSS}}^{Z}$ in power. The differences for the gamma distribution are larger than those for the lognormal distribution. The sizes and powers of the imperfect ranking are reported in Tables D1–D4, and their results are the same.

6 Conclusion

In this paper, we present a novel approach to EL for RSS that leverages the ranking structure and information of the RSS, and the EL test for testing the population mean. The asymptotic distribution of the empirical LR statistic is calculated as a chi-square distribution with one degree of freedom. Through numerical study, we demonstrate that our proposed EL test performs comparably to pivot statistics when the data are from a symmetric distribution, and outperforms all existing methods in terms of power when the data are non-normal and skewed.

It is worth to discuss a naive extension of the EL method by Baklizi (2009) for BRSS to URSS. Here, the naive extension implies that the EL with

$$\sum_{h=1}^{H}\sum_{r=1}^{n_h} p_{hr} = 1. \tag{16}$$

The above constraint (16) introduces at least two difficulties for the use of URSS. First, the observations $Y_{[h]r}$ are exchangeable in building EL, and this makes the stratum with more samples over-weighted in the sense that $\sum_{r=1}^{n_h} p_{hr}$ is larger than $1/H$ (which is true by definition). For this reason, many researchers (MacEachern et al., 2004; Zhang et al., 2014; Moon et al., 2022) suggest to use

$$\sum_{h=1}^{H}\sum_{r=1}^{n_h} w_h p_{hr} = 1,$$

where w_h is subjectively chosen depending on sample size and sample within-stratum variances. Unlike this ambiguity of Baklizi (2009), our new EL can apply both balanced and unbalanced RSS. Second, although not showing the details here, the asymptotic distribution of the EL test statistic of Baklizi (2009) with (16) is not $\chi^2(1)$. It is unknown and should be investigated further.

Finally, we would like to address that the method in this paper could be extended to testing more sophisticated hypotheses, such as those involving the area under the receiver operating characteristic curve and variance, by using the idea of the JEL in Section 5. Although EL has the advantage of not requiring any underlying distributional assumptions, only requiring i.i.d samples, the nonlinear constraints of more complex statistics can present computational difficulties. The JEL method addresses these difficulties by treating jackknife pseudo-values as i.i.d. samples, linearizing nonlinear constraints and transforming the problem to a standard EL testing problem. We leave further exploration of this extension as a future work.

Acknowledgements

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2021R1A6A1A10044950, NRF-2021R1A2C1010786) and Learning and Academic Research Institution for Master's, PhD students and Postdocs (LAMP) Program of the National Research Foundation of Korea (NRF) grant funded by the Ministry of Education (No. RS-2023-00285390).

Data Availability Statement

The data that support the findings of this study are available from the corresponding author (Johan Lim) upon a reasonable request.

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Appendix A: Proof of Theorem 1

Theorem 2. Assume that (i) $\mathbb{E}(Y)=\mu$, (ii) $\mathbb{E}|Y|^3 < \infty$, and (iii) $n_h/n \to q_h$ as $n\to\infty$ for every $h = 1, 2, \cdots, H$ where $n=\sum_{h=1}^{H} n_h$. For fixed H, as $n\to\infty$,

$$\mathrm{EL}_{\mathrm{RSS}}(\mu) \xrightarrow{d} \chi_1^2.$$

Proof. Under the assumptions of Theorem 1, we derive the asymptotic distribution of the empirical log-likelihood ratio statistic $\mathrm{EL}_{\mathrm{RSS}}(\mu) = -2\log \mathcal{R}(\mu)$.

Let $z_{hr} = v\left(Y_{[h],r} - \mu_h\right)/\lambda_h$. Then, we can rewrite p_{hr} in (6) and λ_h in (7) as

$$p_{rh} = \frac{1}{\lambda_h(1+z_{hr})}, \quad h=1,2,\cdots,H, \quad r=1,2,\cdots,n_h,$$

and

$$\lambda_h = H\sum_{r=1}^{n_h}\frac{1}{1+z_{hr}}, \qquad h=1,2,\cdots,H,$$

respectively. Further,

$$-2\log R(\mu) = 2\sum_{h=1}^{H}\sum_{r=1}^{n_h}\left\{\log(1+z_{hr}) + \log\left(\frac{1}{n_h}\sum_{r'=1}^{n_h}\frac{1}{1+z_{hr'}}\right)\right\}.$$

Using the Taylor series expansion, we have

$$\begin{aligned}
\log\left(\frac{1}{n_h}\sum_{r'=1}^{n_h}\frac{1}{1+z_{hr'}}\right) &= \log\left\{\frac{1}{n_h}\sum_{r'=1}^{n_h}\left(1 - z_{hr'} + z_{hr'}^2 + O_p\left(z_{hr'}^3\right)\right)\right\} \\
&= \log\left\{1 - \overline{z_h} + \overline{z_h^2} + O_p\left(\overline{z_h^3}\right)\right\} \\
&= -\left\{\overline{z_h} - \overline{z_h^2} + O_p\left(\overline{z_h^3}\right)\right\} \\
&\quad -\frac{1}{2}\left\{\overline{z_h} - \overline{z_h^2} + O_p\left(\overline{z_h^3}\right)\right\}^2 \\
&\quad + O_p\left(\left\{\overline{z_h} - \overline{z_h^2} + O_p\left(\overline{z_h^3}\right)\right\}^2\right) \\
&= -\overline{z_h} + \overline{z_h^2} + O_p\left(\overline{z_h}2 + \overline{z_h}\cdot\overline{z_h^2} + \left(\overline{z_h^2}\right)^2 + \overline{z_h^3}\right),
\end{aligned}$$

where

$$\overline{z_h^k} = \frac{1}{n_h}\sum_{r'=1}^{n_h} z_{hr'}^k, \quad k=1,2,3.$$

Then,

$$
\begin{aligned}
-2\log R(\mu) &= 2\sum_{h=1}^{H}\sum_{r=1}^{n_h}\left\{\log(1+z_{hr})+\log\left(\frac{1}{n_h}\sum_{r'=1}^{n_h}\frac{1}{1+z_{hr'}}\right)\right\}\\
&= 2\sum_{h=1}^{H}\sum_{r=1}^{n_h}\left\{z_{hr}-\frac{1}{2}z_{hr}^2+O_p\left(z_{hr}^3\right)-\overline{z_h}+\overline{z_h^2}+O_p\left(\overline{z_h}2+\overline{z_h}\cdot\overline{z_h^2}+\left(\overline{z_h^2}\right)^2+\overline{z_h^3}\right)\right\}\\
&= \sum_{h=1}^{H}\sum_{r=1}^{n_h}z_{hr}^2+\sum_{h=1}^{H}\sum_{r=1}^{n_h}O_p\left(\overline{z_h}2+\overline{z_h}\cdot\overline{z_h^2}+\left(\overline{z_h^2}\right)^2+\overline{z_h^3}\right)\\
&= \sum_{h=1}^{H}\sum_{r=1}^{n_h}z_{hr}^2+O_p\left(\sum_{h=1}^{H}\frac{1}{n_h}\left(\sum_{r=1}^{n_h}z_{hr}\right)^2\right)+O_p\left(\sum_{h=1}^{H}\frac{1}{n_h}\left(\sum_{r=1}^{n_h}z_{hr}\right)\left(\sum_{r'=1}^{n_h}z_{hr'}^2\right)\right)\\
&\quad+O_p\left(\sum_{h=1}^{H}\frac{1}{n_h}\left(\sum_{r=1}^{n_h}z_{hr}^2\right)^2\right)+O_p\left(\sum_{h=1}^{H}\sum_{r=1}^{n_h}z_{hr}^3\right).
\end{aligned}
\tag{17}
$$

To compute the asymptotic distribution of each term, we find the approximations for z_{hr}, λ_h, and v as follows.

From equation (7), we have

$$
\begin{aligned}
\lambda_h = H\sum_{r=1}^{n_h}\frac{1}{1+z_{hr}} &= H\sum_{r=1}^{n_h}\left(1-z_{hr}+O_p\left(z_{hr}^2\right)\right)\\
&= Hn_h\left(1-\overline{z_h}+O_p\left(\overline{z_h^2}\right)\right)\approx Hn_h\left(1+o_p(1)\right).
\end{aligned}
$$

Using z_{hr} and the Taylor series expansion, equation (8) can be written as

$$
\begin{aligned}
0 &= \sum_{h=1}^{H}\sum_{r=1}^{n_h}\frac{Y_{[h],r}-\mu_h}{\lambda_h+v\left(Y_{[h],r}-\mu_h\right)}\\
&= \sum_{h=1}^{H}\sum_{r=1}^{n_h}\frac{Y_{[h],r}-\mu_h}{\lambda_h}\frac{1}{1+z_{hr}}\\
&= \sum_{h=1}^{H}\sum_{r=1}^{n_h}\left(\frac{Y_{[h],r}-\mu_h}{\lambda_h}\right)\left\{1-z_{hr}+O_p\left(z_{hr}^2\right)\right\}\\
&= \sum_{h=1}^{H}\sum_{r=1}^{n_h}\frac{Y_{[h],r}-\mu_h}{\lambda_h}\left\{1+O_p\left(z_{hr}^2\right)\right\}-\sum_{h=1}^{H}\sum_{r=1}^{n_h}\left(\frac{Y_{[h],r}-\mu_h}{\lambda_h}\right)^2 v.
\end{aligned}
$$

Then,

$$
\begin{aligned}
v &= \left\{\sum_{h=1}^{H}\sum_{r=1}^{n_h}\frac{Y_{[h],r}-\mu_h}{\lambda_h}\left(1+O_p\left(z_{hr}^2\right)\right)\right\}\Big/\left\{\sum_{h=1}^{H}\sum_{r=1}^{n_h}\left(\frac{Y_{[h],r}-\mu_h}{\lambda_h}\right)^2\right\}\\
&= \left\{\sum_{h=1}^{H}\sum_{r=1}^{n_h}\frac{Y_{[h],r}-\mu_h}{\lambda_h}\left(1+O_p\left(z_{hr}^2\right)\right)\right\}\Big/\left\{\sum_{h=1}^{H}\sum_{r=1}^{n_h}\left(\frac{Y_{[h],r}-\mu_h}{\lambda_h}\right)^2\right\}^{1/2}\\
&\quad\times\left\{\sum_{h=1}^{H}\sum_{r=1}^{n_h}\left(\frac{Y_{[h],r}-\mu_h}{\lambda_h}\right)^2\right\}^{-1/2}\\
&\approx O_p(1)H\left(\sum_{h=1}^{H}\frac{1}{n_h}\right)^{-1/2}\left(1+o_p(1)\right)
\end{aligned}
$$

and

$$z_{hr} = v\frac{Y_{[h],\,r} \; - \; \mu_h}{\lambda_h} = O_p\left(n_h^{-1}\left(\sum_{h'=1}^{H}\frac{1}{n_{h'}}\right)^{-1/2}\right).$$

By the formula of v, the first term in (17) is

$$\begin{aligned}
\sum_{h=1}^{H}\sum_{r=1}^{n_h} z_{hr}^2 &= \sum_{h=1}^{H}\sum_{r=1}^{n_h}\frac{v^2}{\lambda_h^2}\left(Y_{[h],\,r} \; - \; \mu_h\right)^2 \\
&= \sum_{h=1}^{H}\sum_{r=1}^{n_h} v^2\left(\frac{Y_{[h],\,r} \; - \; \mu_h}{\lambda_h}\right)^2 \\
&= \left\{\sum_{h=1}^{H}\sum_{r=1}^{n_h}\frac{Y_{[h],\,r} \; - \; \mu_h}{\lambda_h}\left(1 + O_p\left(z_{hr}^2\right)\right)\right\}^2 \Bigg/ \left\{\sum_{h=1}^{H}\sum_{r=1}^{n_h}\left(\frac{Y_{[h],\,r} \; - \; \mu_h}{\lambda_h}\right)^2\right\}^2 \\
&\quad \times \sum_{h=1}^{H}\sum_{r=1}^{n_h}\left(\frac{Y_{[h],\,r} \; - \; \mu_h}{\lambda_h}\right)^2 \\
&= \left\{\sum_{h=1}^{H}\sum_{r=1}^{n_h}\left(\frac{Y_{[h],\,r} \; - \; \mu_h}{\lambda_h}\right)^2\right\}^{-1} \times \left\{\sum_{h=1}^{H}\sum_{r=1}^{n_h}\frac{Y_{[h],\,r} \; - \; \mu_h}{\lambda_h}\left(1 + O_p\left(z_{hr}^2\right)\right)\right\}^2 \\
&\approx \left\{\sum_{h=1}^{H}\frac{1}{n_h}\sum_{r=1}^{n_h}\left(Y_{[h],\,r} \; - \; \mu_h\right)\right\}^2 \Bigg/ \left\{\sum_{h=1}^{H}\frac{1}{n_h^2}\sum_{r=1}^{n_h}\left(Y_{[h],\,r} \; - \; \mu_h\right)^2\right\}\left(1 + o_p(1)\right)
\end{aligned}$$

and has the same asymptotic distribution with

$$\mathrm{A} = \frac{n\left\{\sum_{h=1}^{H}\frac{1}{n_h}\sum_{r=1}^{n_h}\left(Y_{[h],\,r} \; - \; \mu_h\right)\right\}^2}{n\sum_{h=1}^{H}\frac{1}{n_h^2}\sum_{r=1}^{n_h}\left(Y_{[h],\,r} \; - \; \mu_h\right)^2} := \frac{\mathrm{A}_{\mathrm{num}}}{\mathrm{A}_{\mathrm{den}}},$$

where $\mathrm{A}_{\mathrm{num}} = \left\{\sqrt{n}\sum_{h=1}^{H}\frac{1}{n_h}\sum_{r=1}^{n_h}\left(Y_{[h],\,r} \; - \; \mu_h\right)\right\}^2$ and $\mathrm{A}_{\mathrm{den}} = n\sum_{h=1}^{H}\frac{1}{n_h^2}\sum_{r=1}^{n_h}\left(Y_{[h],\,r} \; - \; \mu_h\right)^2$.

Let, for $h = 1, 2, \ldots, H$, $U_h = \frac{1}{\sqrt{n_h}\sigma_h}\sum_{h=1}^{n_h}\left(Y_{[h],\,r} \; - \; \mu_h\right)$ and $V_h = \frac{1}{n_h}\sum_{r=1}^{n_h}\left(Y_{[h],\,r} \; - \; \mu_h\right)^2$, which converges in distribution to the standard normal distribution and in probability to σ_h^2, as $n_h \rightarrow \infty$, respectively. Then, we have

$$\sum_{h=1}^{H}\frac{\sqrt{n}\sigma_h}{\sqrt{n_h}}U_h \rightarrow N\left(0, \; \sum_{h=1}^{H}\frac{\sigma_h^2}{q_h}\right)$$

where $\lim_{n\rightarrow\infty} n_h/n = q_h \; > \; 0$. In consequence,

$$\mathrm{A}_{\mathrm{num}} = \left\{\sum_{h=1}^{H}\frac{\sqrt{n}\sigma_h}{\sqrt{n_h}}U_h\right\}^2 \xrightarrow{\mathrm{d}} \sum_{h=1}^{H}\frac{\sigma_h^2}{q_h}\cdot\chi_1^2.$$

Next,

$$\mathrm{A}_{\mathrm{den}} = \sum_{h=1}^{H}\frac{n}{n_h}V_h \xrightarrow{\mathrm{p}} \sum_{h=1}^{H}\frac{\sigma_h^2}{q_h}.$$

By Slutsky's theorem, $A = A_{num}/A_{den}$ converges in distribution to the χ_1^2.

Lastly, by the approximations for z_{hr}, λ_h and v, we obtain that the other terms in (17) are $O_p(n^{-1})$. Therefore, we establish the asymptotic distribution:

$$-2\log R(\mu) \xrightarrow{d} \chi_1^2.$$

Appendix B: Results of numerical comparison

Table B1. *With BRSS under normal distribution, approximated size and power of four RSS-based testing methods (Pivot,* S^+_{RSS}*,* EL^B_{RSS} *and* EL_{RSS}*) and SRS-based testing method (*EL_{SRS}*).*

ρ	H	n_h	δ	EL_{SRS}	*Pivot*	S^+_{RSS}	EL^B_{RSS}	EL_{RSS}
1	2	(8,8)	0	0.076(0.050)	0.067(0.050)	0.041(0.041)	0.101(0.050)	0.090(0.050)
			0.25	0.211(0.161)	0.245(0.203)	0.126(0.126)	0.298(0.200)	0.284(0.203)
			0.50	0.529(0.456)	0.686(0.627)	0.404(0.404)	0.738(0.618)	0.719(0.619)
		(16,16)	0	0.068(0.050)	0.065(0.050)	0.064(0.026)	0.077(0.050)	0.069(0.050)
			0.25	0.307(0.254)	0.414(0.366)	0.296(0.164)	0.448(0.356)	0.425(0.361)
			0.50	0.799(0.757)	0.930(0.911)	0.764(0.610)	0.938(0.905)	0.934(0.907)
	4	(8,8,8,8)	0	0.061(0.050)	0.058(0.050)	0.034(0.034)	0.070(0.050)	0.074(0.050)
			0.25	0.294(0.272)	0.583(0.555)	0.269(0.269)	0.611(0.551)	0.627(0.553)
			0.50	0.807(0.784)	0.991(0.988)	0.786(0.786)	0.991(0.987)	0.992(0.988)
		(16,16,16,16)	0	0.051(0.050)	0.058(0.050)	0.061(0.032)	0.064(0.050)	0.065(0.050)
			0.25	0.514(0.507)	0.862(0.844)	0.609(0.474)	0.869(0.841)	0.872(0.838)
			0.50	0.976(0.974)	1.000(1.000)	0.991(0.977)	1.000(1.000)	1.000(1.000)
0.9	2	(8,8)	0	0.076(0.050)	0.071(0.050)	0.050(0.050)	0.103(0.050)	0.096(0.050)
			0.25	0.211(0.161)	0.243(0.190)	0.143(0.143)	0.299(0.190)	0.280(0.180)
			0.50	0.529(0.456)	0.654(0.578)	0.402(0.402)	0.717(0.576)	0.698(0.553)
		(16,16)	0	0.068(0.050)	0.064(0.050)	0.075(0.028)	0.078(0.050)	0.068(0.050)
			0.25	0.307(0.254)	0.385(0.340)	0.294(0.176)	0.419(0.336)	0.394(0.334)
			0.50	0.799(0.757)	0.909(0.881)	0.751(0.605)	0.927(0.876)	0.916(0.878)
	4	(8,8,8,8)	0	0.061(0.050)	0.060(0.050)	0.050(0.050)	0.072(0.050)	0.076(0.050)
			0.25	0.294(0.272)	0.497(0.469)	0.279(0.279)	0.536(0.461)	0.543(0.460)
			0.50	0.807(0.784)	0.970(0.963)	0.766(0.766)	0.975(0.961)	0.975(0.963)
		(16,16,16,16)	0	0.051(0.050)	0.057(0.050)	0.086(0.046)	0.065(0.050)	0.063(0.050)
			0.25	0.514(0.507)	0.786(0.768)	0.608(0.484)	0.799(0.766)	0.800(0.766)
			0.50	0.976(0.974)	1.000(1.000)	0.988(0.972)	1.000(1.000)	1.000(1.000)
0.7	2	(8,8)	0	0.076(0.050)	0.077(0.050)	0.063(0.015)	0.113(0.050)	0.100(0.050)
			0.25	0.211(0.161)	0.217(0.168)	0.156(0.057)	0.282(0.161)	0.253(0.162)
			0.50	0.529(0.456)	0.590(0.514)	0.409(0.202)	0.659(0.510)	0.629(0.487)
		(16,16)	0	0.068(0.050)	0.058(0.050)	0.085(0.035)	0.071(0.050)	0.062(0.050)
			0.25	0.307(0.254)	0.351(0.327)	0.303(0.185)	0.389(0.330)	0.362(0.327)
			0.50	0.799(0.757)	0.876(0.863)	0.763(0.617)	0.897(0.861)	0.883(0.858)
	4	(8,8,8,8)	0	0.061(0.050)	0.055(0.050)	0.074(0.030)	0.070(0.050)	0.069(0.050)
			0.25	0.294(0.272)	0.397(0.387)	0.314(0.184)	0.445(0.386)	0.440(0.386)
			0.50	0.807(0.784)	0.914(0.907)	0.759(0.607)	0.933(0.907)	0.933(0.907)
		(16,16,16,16)	0	0.051(0.050)	0.057(0.050)	0.125(0.039)	0.065(0.050)	0.064(0.050)
			0.25	0.514(0.507)	0.667(0.647)	0.599(0.364)	0.689(0.643)	0.684(0.645)
			0.50	0.976(0.974)	0.997(0.997)	0.977(0.921)	0.998(0.997)	0.998(0.997)

The size of tests is $\alpha = 0.05$. The parentheses are the empirically corrected powers for each method.

Table B2. *With URSS under normal distribution, approximated size and power of three RSS-based testing methods (Pivot,* S^{+}_{RSS} *and* EL_{RSS}*) and SRS-based testing method (*EL_{SRS}*).*

				$\rho = 1$			$\rho = 0.9$			$\rho = 0.7$		
H	n_h	δ	EL_{SRS}	*Pivot*	S^{+}_{RSS}	EL_{RSS}	*Pivot*	S^{+}_{RSS}	EL_{RSS}	*Pivot*	S^{+}_{RSS}	EL_{RSS}
2	(6,10)	0	0.076(0.050)	0.076(0.050)	0.043(0.043)	0.104(0.050)	0.082(0.050)	0.051(0.013)	0.110(0.050)	0.076(0.050)	0.067(0.020)	0.101(0.050)
		0.25	0.211(0.161)	0.255(0.188)	0.112(0.112)	0.296(0.188)	0.226(0.170)	0.112(0.032)	0.275(0.152)	0.211(0.157)	0.103(0.031)	0.255(0.152)
		0.50	0.529(0.456)	0.644(0.555)	0.352(0.352)	0.697(0.543)	0.623(0.521)	0.350(0.143)	0.680(0.500)	0.571(0.488)	0.324(0.129)	0.632(0.472)
	(12,20)	0	0.068(0.050)	0.060(0.050)	0.059(0.022)	0.067(0.050)	0.067(0.050)	0.077(0.034)	0.071(0.050)	0.066(0.050)	0.103(0.046)	0.068(0.050)
		0.25	0.307(0.254)	0.288(0.262)	0.254(0.133)	0.305(0.260)	0.374(0.327)	0.247(0.138)	0.387(0.327)	0.341(0.300)	0.209(0.112)	0.360(0.304)
		0.50	0.799(0.757)	0.909(0.891)	0.725(0.569)	0.915(0.882)	0.882(0.855)	0.705(0.533)	0.888(0.850)	0.841(0.808)	0.629(0.464)	0.847(0.806)
	(10,6)	0	0.076(0.050)	0.073(0.050)	0.038(0.038)	0.094(0.050)	0.077(0.050)	0.052(0.012)	0.100(0.050)	0.074(0.050)	0.065(0.017)	0.100(0.050)
		0.25	0.211(0.161)	0.240(0.174)	0.139(0.139)	0.290(0.180)	0.234(0.175)	0.166(0.063)	0.277(0.173)	0.210(0.169)	0.209(0.087)	0.260(0.162)
		0.50	0.529(0.456)	0.660(0.572)	0.445(0.445)	0.713(0.567)	0.621(0.539)	0.468(0.241)	0.676(0.523)	0.561(0.497)	0.504(0.282)	0.621(0.482)
	(20,12)	0	0.068(0.050)	0.062(0.050)	0.065(0.028)	0.072(0.050)	0.065(0.050)	0.077(0.031)	0.074(0.050)	0.065(0.050)	0.101(0.044)	0.073(0.050)
		0.25	0.307(0.254)	0.390(0.348)	0.303(0.176)	0.407(0.352)	0.350(0.309)	0.338(0.214)	0.369(0.306)	0.321(0.276)	0.400(0.269)	0.336(0.272)
		0.50	0.799(0.757)	0.912(0.890)	0.782(0.651)	0.919(0.890)	0.893(0.868)	0.792(0.673)	0.900(0.862)	0.849(0.816)	0.830(0.723)	0.860(0.811)
4	(6,6,10,10)	0	0.061(0.050)	0.065(0.050)	0.055(0.019)	0.090(0.050)	0.060(0.050)	0.077(0.028)	0.081(0.050)	0.064(0.050)	0.138(0.067)	0.087(0.050)
		0.25	0.294(0.272)	0.558(0.508)	0.309(0.158)	0.603(0.503)	0.475(0.449)	0.279(0.142)	0.531(0.449)	0.388(0.355)	0.233(0.123)	0.437(0.340)
		0.50	0.807(0.784)	0.982(0.975)	0.820(0.642)	0.986(0.973)	0.958(0.95)	0.778(0.603)	0.967(0.949)	0.892(0.872)	0.678(0.515)	0.915(0.865)
	(12,12,20,20)	0	0.051(0.050)	0.058(0.050)	0.065(0.032)	0.064(0.050)	0.053(0.050)	0.092(0.050)	0.062(0.050)	0.056(0.050)	0.169(0.062)	0.062(0.050)
		0.25	0.514(0.507)	0.841(0.822)	0.586(0.440)	0.849(0.818)	0.757(0.749)	0.501(0.371)	0.770(0.744)	0.642(0.623)	0.385(0.184)	0.657(0.622)
		0.50	0.976(0.974)	1.000(1.000)	0.989(0.973)	1.000(1.000)	0.999(0.999)	0.974(0.939)	0.999(0.999)	0.995(0.995)	0.925(0.782)	0.996(0.995)
	(10,10,6,6)	0	0.061(0.050)	0.056(0.050)	0.048(0.048)	0.081(0.050)	0.063(0.050)	0.082(0.030)	0.083(0.050)	0.064(0.050)	0.137(0.021)	0.084(0.050)
		0.25	0.294(0.272)	0.565(0.543)	0.376(0.376)	0.626(0.541)	0.481(0.426)	0.432(0.272)	0.537(0.431)	0.376(0.328)	0.519(0.230)	0.428(0.333)
		0.50	0.807(0.784)	0.988(0.985)	0.884(0.884)	0.992(0.985)	0.955(0.937)	0.885(0.792)	0.967(0.939)	0.887(0.865)	0.904(0.702)	0.912(0.868)
	(20,20,12,12)	0	0.051(0.050)	0.054(0.050)	0.060(0.027)	0.060(0.050)	0.055(0.050)	0.092(0.025)	0.061(0.050)	0.054(0.050)	0.168(0.029)	0.062(0.050)
		0.25	0.514(0.507)	0.841(0.833)	0.622(0.498)	0.851(0.834)	0.757(0.745)	0.697(0.464)	0.772(0.737)	0.633(0.623)	0.783(0.479)	0.654(0.622)
		0.50	0.976(0.974)	1.000(1.000)	0.994(0.986)	1.000(1.000)	1.000(1.000)	0.993(0.968)	1.000(1.000)	0.995(0.994)	0.992(0.950)	0.995(0.993)

The size of tests is $\alpha = 0.05$. The parentheses are the empirically corrected powers for each method.

Table B3. *With BRSS under t distribution, approximated size and power of four RSS-based testing methods (Pivot, S^{+}_{RSS}, EL^{B}_{RSS} and EL_{RSS}) and SRS-based testing method (EL_{SRS}).*

ρ	H	n_h	δ	EL_{SRS}	*Pivot*	S^{+}_{RSS}	EL^{B}_{RSS}	EL_{RSS}
1	2	(8,8)	0	0.086(0.050)	0.068(0.050)	0.043(0.043)	0.111(0.050)	0.102(0.050)
			0.25	0.187(0.128)	0.194(0.161)	0.121(0.121)	0.260(0.157)	0.248(0.157)
			0.50	0.415(0.327)	0.512(0.456)	0.358(0.358)	0.579(0.434)	0.566(0.432)
		(16,16)	0	0.063(0.050)	0.057(0.050)	0.062(0.024)	0.077(0.050)	0.070(0.050)
			0.25	0.241(0.216)	0.289(0.271)	0.265(0.145)	0.326(0.248)	0.308(0.252)
			0.50	0.613(0.580)	0.760(0.743)	0.722(0.556)	0.780(0.707)	0.757(0.703)
	4	(8,8,8,8)	0	0.064(0.050)	0.058(0.050)	0.036(0.036)	0.077(0.050)	0.086(0.050)
			0.25	0.233(0.198)	0.400(0.378)	0.232(0.232)	0.438(0.364)	0.449(0.360)
			0.50	0.616(0.572)	0.884(0.875)	0.736(0.736)	0.888(0.847)	0.894(0.845)
		(16,16,16,16)	0	0.065(0.050)	0.048(0.050)	0.055(0.024)	0.060(0.050)	0.063(0.050)
			0.25	0.359(0.319)	0.635(0.638)	0.573(0.430)	0.642(0.614)	0.641(0.607)
			0.50	0.856(0.829)	0.984(0.984)	0.978(0.961)	0.983(0.980)	0.976(0.972)
0.9	2	(8,8)	0	0.086(0.050)	0.064(0.050)	0.053(0.013)	0.107(0.050)	0.097(0.050)
			0.25	0.187(0.128)	0.181(0.143)	0.136(0.047)	0.246(0.144)	0.232(0.142)
			0.50	0.415(0.327)	0.486(0.437)	0.373(0.177)	0.547(0.415)	0.538(0.416)
		(16,16)	0	0.063(0.050)	0.061(0.050)	0.080(0.029)	0.087(0.050)	0.077(0.050)
			0.25	0.241(0.216)	0.270(0.235)	0.282(0.162)	0.302(0.223)	0.280(0.222)
			0.50	0.613(0.580)	0.725(0.692)	0.704(0.543)	0.743(0.667)	0.722(0.653)
	4	(8,8,8,8)	0	0.064(0.050)	0.062(0.050)	0.063(0.019)	0.09(0.050)	0.093(0.050)
			0.25	0.233(0.198)	0.328(0.297)	0.252(0.139)	0.368(0.277)	0.377(0.281)
			0.50	0.616(0.572)	0.816(0.793)	0.715(0.547)	0.830(0.763)	0.838(0.765)
		(16,16,16,16)	0	0.065(0.050)	0.049(0.050)	0.095(0.025)	0.065(0.050)	0.063(0.050)
			0.25	0.359(0.319)	0.539(0.540)	0.562(0.327)	0.555(0.524)	0.554(0.517)
			0.50	0.856(0.829)	0.974(0.974)	0.974(0.906)	0.972(0.965)	0.967(0.957)
0.7	2	(8,8)	0	0.086(0.050)	0.068(0.050)	0.061(0.015)	0.113(0.050)	0.101(0.050)
			0.25	0.187(0.128)	0.174(0.142)	0.152(0.052)	0.244(0.137)	0.224(0.141)
			0.50	0.415(0.327)	0.443(0.392)	0.384(0.186)	0.517(0.376)	0.498(0.378)
		(16,16)	0	0.063(0.050)	0.059(0.050)	0.091(0.042)	0.082(0.050)	0.073(0.050)
			0.25	0.241(0.216)	0.263(0.235)	0.303(0.182)	0.299(0.225)	0.276(0.214)
			0.50	0.613(0.580)	0.674(0.650)	0.690(0.546)	0.699(0.622)	0.671(0.599)
	4	(8,8,8,8)	0	0.064(0.050)	0.057(0.050)	0.083(0.035)	0.084(0.050)	0.082(0.050)
			0.25	0.233(0.198)	0.269(0.253)	0.277(0.155)	0.320(0.243)	0.318(0.239)
			0.50	0.616(0.572)	0.728(0.714)	0.702(0.547)	0.758(0.685)	0.758(0.682)
		(16,16,16,16)	0	0.065(0.050)	0.051(0.050)	0.130(0.038)	0.064(0.050)	0.060(0.050)
			0.25	0.359(0.319)	0.457(0.455)	0.556(0.337)	0.476(0.444)	0.471(0.441)
			0.50	0.856(0.829)	0.934(0.933)	0.960(0.879)	0.935(0.924)	0.928(0.915)

The size of tests is $\alpha = 0.05$. The parentheses are the empirically corrected powers for each method.

Table B4. *With URSS under t distribution, approximated size and power of three RSS-based testing methods (Pivot,* S^{+}_{RSS}*, and* EL_{RSS}*) and SRS-based testing method (*EL_{SRS}*).*

				$\rho = 1$			$\rho = 0.9$			$\rho = 0.7$		
H	n_h	δ	EL_{SRS}	*Pivot*	S^{+}_{RSS}	EL_{RSS}	*Pivot*	S^{+}_{RSS}	EL_{RSS}	*Pivot*	S^{+}_{RSS}	EL_{RSS}
2	(6,10)	0	0.086(0.050)	0.074(0.050)	0.041(0.041)	0.118(0.050)	0.068(0.050)	0.054(0.012)	0.112(0.050)	0.067(0.050)	0.067(0.016)	0.109(0.050)
		0.25	0.187(0.128)	0.208(0.164)	0.113(0.113)	0.270(0.149)	0.201(0.161)	0.106(0.029)	0.259(0.140)	0.182(0.144)	0.099(0.029)	0.232(0.131)
		0.50	0.415(0.327)	0.504(0.445)	0.316(0.316)	0.579(0.409)	0.487(0.431)	0.312(0.125)	0.551(0.401)	0.449(0.384)	0.282(0.116)	0.517(0.366)
	(12,20)	0	0.063(0.050)	0.057(0.050)	0.062(0.022)	0.074(0.050)	0.065(0.050)	0.086(0.033)	0.083(0.050)	0.067(0.050)	0.115(0.022)	0.078(0.050)
		0.25	0.241(0.216)	0.301(0.286)	0.578(0.414)	0.317(0.268)	0.272(0.227)	0.231(0.122)	0.291(0.213)	0.251(0.198)	0.195(0.044)	0.272(0.190)
		0.50	0.613(0.580)	0.731(0.714)	0.692(0.518)	0.728(0.673)	0.699(0.650)	0.635(0.470)	0.697(0.608)	0.662(0.609)	0.565(0.260)	0.663(0.565)
	(10,6)	0	0.086(0.050)	0.075(0.050)	0.041(0.041)	0.116(0.050)	0.070(0.050)	0.048(0.048)	0.112(0.050)	0.071(0.050)	0.063(0.017)	0.114(0.050)
		0.25	0.187(0.128)	0.174(0.125)	0.137(0.137)	0.234(0.131)	0.164(0.127)	0.159(0.159)	0.225(0.127)	0.161(0.114)	0.200(0.083)	0.218(0.124)
		0.50	0.415(0.327)	0.493(0.405)	0.388(0.388)	0.562(0.395)	0.450(0.386)	0.418(0.418)	0.522(0.366)	0.428(0.353)	0.475(0.273)	0.494(0.340)
	(20,12)	0	0.063(0.050)	0.057(0.050)	0.062(0.024)	0.076(0.050)	0.062(0.050)	0.089(0.039)	0.082(0.050)	0.064(0.050)	0.118(0.022)	0.083(0.050)
		0.25	0.241(0.216)	0.273(0.249)	0.291(0.166)	0.304(0.235)	0.238(0.208)	0.345(0.217)	0.269(0.200)	0.222(0.196)	0.400(0.161)	0.255(0.184)
		0.50	0.613(0.580)	0.749(0.725)	0.743(0.599)	0.755(0.676)	0.703(0.667)	0.766(0.629)	0.711(0.623)	0.654(0.615)	0.800(0.537)	0.664(0.568)
4	(6,6,10,10)	0	0.064(0.050)	0.063(0.050)	0.052(0.014)	0.095(0.050)	0.062(0.050)	0.090(0.035)	0.093(0.050)	0.059(0.050)	0.150(0.072)	0.089(0.050)
		0.25	0.233(0.198)	0.401(0.363)	0.289(0.142)	0.459(0.340)	0.338(0.304)	0.259(0.124)	0.387(0.285)	0.275(0.251)	0.217(0.110)	0.327(0.233)
		0.50	0.616(0.572)	0.857(0.838)	0.775(0.594)	0.878(0.816)	0.789(0.764)	0.694(0.505)	0.821(0.735)	0.692(0.667)	0.600(0.430)	0.729(0.635)
	(12,12,20,20)	0	0.065(0.050)	0.053(0.050)	0.062(0.026)	0.068(0.050)	0.053(0.050)	0.109(0.028)	0.066(0.050)	0.056(0.050)	0.200(0.073)	0.069(0.050)
		0.25	0.359(0.319)	0.614(0.605)	0.541(0.397)	0.613(0.560)	0.525(0.518)	0.437(0.211)	0.533(0.486)	0.438(0.427)	0.324(0.155)	0.447(0.396)
		0.50	0.856(0.829)	0.973(0.971)	0.980(0.950)	0.963(0.953)	0.959(0.958)	0.944(0.812)	0.950(0.937)	0.920(0.917)	0.871(0.702)	0.915(0.890)
	(10,10,6,6)	0	0.064(0.050)	0.067(0.050)	0.055(0.018)	0.099(0.050)	0.061(0.050)	0.089(0.033)	0.093(0.050)	0.062(0.050)	0.142(0.029)	0.089(0.050)
		0.25	0.233(0.198)	0.358(0.309)	0.337(0.195)	0.426(0.303)	0.300(0.278)	0.406(0.250)	0.357(0.262)	0.249(0.224)	0.509(0.223)	0.303(0.215)
		0.50	0.616(0.572)	0.878(0.846)	0.836(0.698)	0.895(0.813)	0.808(0.787)	0.857(0.737)	0.843(0.763)	0.708(0.679)	0.883(0.648)	0.751(0.655)
	(20,20,12,12)	0	0.065(0.050)	0.051(0.050)	0.060(0.025)	0.067(0.050)	0.052(0.050)	0.112(0.028)	0.064(0.050)	0.052(0.050)	0.187(0.038)	0.064(0.050)
		0.25	0.359(0.319)	0.615(0.612)	0.595(0.460)	0.636(0.591)	0.508(0.498)	0.673(0.433)	0.534(0.486)	0.427(0.422)	0.772(0.467)	0.452(0.411)
		0.50	0.856(0.829)	0.990(0.990)	0.987(0.970)	0.987(0.982)	0.975(0.974)	0.986(0.952)	0.970(0.962)	0.935(0.932)	0.990(0.935)	0.936(0.918)

The size of tests is $\alpha = 0.05$. The parentheses are the empirically corrected powers for each method.

Table B5. *Under gamma distribution, approximated size and power of three RSS-based testing methods (Pivot, EL_{RSS}^{B} and EL_{RSS}) and SRS-based testing method (EL_{SRS}).*

				$\rho = 1$			$\rho = 0.9$			$\rho = 0.7$		
H	n_h	δ	EL_{SRS}	*Pivot*	EL_{RSS}^{B}	EL_{RSS}	*Pivot*	EL_{RSS}^{B}	EL_{RSS}	*Pivot*	EL_{RSS}^{B}	EL_{RSS}
2	(8,8)	0	0.095(0.050)	0.088(0.050)	0.119(0.050)	0.120(0.050)	0.084(0.050)	0.114(0.050)	0.108(0.050)	0.077(0.050)	0.109(0.050)	0.100(0.050)
		0.25	0.117(0.063)	0.099(0.055)	0.166(0.070)	0.160(0.068)	0.097(0.056)	0.167(0.078)	0.149(0.068)	0.089(0.058)	0.160(0.075)	0.137(0.067)
		0.50	0.266(0.165)	0.271(0.166)	0.395(0.226)	0.380(0.215)	0.234(0.151)	0.372(0.210)	0.343(0.177)	0.216(0.156)	0.343(0.202)	0.308(0.180)
	(16,16)	0	0.067(0.050)	0.062(0.050)	0.075(0.050)	0.069(0.050)	0.071(0.050)	0.087(0.050)	0.078(0.050)	0.064(0.050)	0.078(0.050)	0.069(0.050)
		0.25	0.147(0.107)	0.136(0.108)	0.201(0.140)	0.187(0.150)	0.133(0.092)	0.195(0.126)	0.179(0.129)	0.121(0.092)	0.189(0.121)	0.163(0.124)
		0.50	0.437(0.361)	0.489(0.431)	0.604(0.509)	0.586(0.523)	0.458(0.379)	0.573(0.461)	0.544(0.461)	0.403(0.340)	0.525(0.422)	0.488(0.422)
4	(8,8,8,8)	0	0.071(0.050)	0.073(0.050)	0.084(0.050)	0.090(0.050)	0.066(0.050)	0.082(0.050)	0.083(0.050)	0.067(0.050)	0.088(0.050)	0.086(0.050)
		0.25	0.152(0.101)	0.207(0.150)	0.274(0.199)	0.286(0.196)	0.183(0.140)	0.248(0.182)	0.240(0.173)	0.150(0.113)	0.218(0.149)	0.207(0.141)
		0.50	0.450(0.359)	0.693(0.611)	0.778(0.695)	0.790(0.687)	0.598(0.529)	0.697(0.606)	0.687(0.586)	0.490(0.413)	0.614(0.501)	0.593(0.477)
	(16,16,16,16)	0	0.053(0.050)	0.058(0.050)	0.062(0.050)	0.061(0.050)	0.058(0.050)	0.063(0.050)	0.063(0.050)	0.061(0.050)	0.067(0.050)	0.065(0.050)
		0.25	0.233(0.229)	0.381(0.356)	0.441(0.403)	0.449(0.412)	0.322(0.297)	0.382(0.338)	0.372(0.330)	0.249(0.217)	0.321(0.266)	0.301(0.257)
		0.50	0.708(0.702)	0.948(0.940)	0.962(0.954)	0.963(0.955)	0.885(0.877)	0.920(0.898)	0.915(0.896)	0.791(0.757)	0.847(0.812)	0.837(0.802)
2	(6,10)	0	0.095(0.050)	0.081(0.050)	–	0.106(0.050)	0.080(0.050)	–	0.106(0.050)	0.077(0.050)	–	0.101(0.050)
		0.25	0.117(0.063)	0.102(0.055)	–	0.159(0.071)	0.103(0.061)	–	0.152(0.073)	0.094(0.054)	–	0.144(0.063)
		0.50	0.266(0.165)	0.272(0.176)	–	0.381(0.219)	0.257(0.169)	–	0.356(0.209)	0.229(0.141)	–	0.324(0.170)
	(12,20)	0	0.067(0.050)	0.064(0.050)	–	0.069(0.050)	0.066(0.050)	–	0.070(0.050)	0.064(0.050)	–	0.071(0.050)
		0.25	0.147(0.107)	0.151(0.120)	–	0.203(0.165)	0.150(0.120)	–	0.199(0.159)	0.128(0.097)	–	0.179(0.135)
		0.50	0.437(0.361)	0.514(0.463)	–	0.605(0.546)	0.483(0.425)	–	0.561(0.505)	0.425(0.366)	–	0.511(0.450)
	(10,6)	0	0.095(0.050)	0.094(0.050)	–	0.140(0.050)	0.094(0.050)	–	0.137(0.050)	0.095(0.050)	–	0.135(0.050)
		0.25	0.117(0.063)	0.099(0.048)	–	0.169(0.066)	0.105(0.050)	–	0.170(0.067)	0.093(0.042)	–	0.146(0.052)
		0.50	0.266(0.165)	0.242(0.129)	–	0.370(0.168)	0.228(0.125)	–	0.333(0.143)	0.209(0.102)	–	0.311(0.117)
	(20,12)	0	0.067(0.050)	0.077(0.050)	–	0.082(0.050)	0.071(0.050)	–	0.079(0.050)	0.071(0.050)	–	0.077(0.050)
		0.25	0.147(0.107)	0.120(0.077)	–	0.174(0.108)	0.117(0.080)	–	0.158(0.108)	0.109(0.078)	–	0.153(0.101)
		0.50	0.437(0.361)	0.435(0.336)	–	0.525(0.422)	0.399(0.314)	–	0.501(0.398)	0.353(0.278)	–	0.451(0.355)
4	(6,6,10,10)	0	0.071(0.050)	0.062(0.050)	–	0.079(0.050)	0.068(0.050)	–	0.087(0.050)	0.068(0.050)	–	0.087(0.050)
		0.25	0.152(0.101)	0.227(0.194)	–	0.304(0.239)	0.188(0.148)	–	0.254(0.179)	0.152(0.126)	–	0.214(0.142)
		0.50	0.45(0.359)	0.734(0.699)	–	0.815(0.758)	0.627(0.559)	–	0.712(0.620)	0.492(0.443)	–	0.597(0.492)

(*Continues*)

Table B5 *(Continued)*

H	n_h	δ	EL_{SRS}	$\rho = 1$			$\rho = 0.9$			$\rho = 0.7$		
				Pivot	EL^{B}_{RSS}	EL_{RSS}	*Pivot*	EL^{B}_{RSS}	EL_{RSS}	*Pivot*	EL^{B}_{RSS}	EL_{RSS}
	(12,12,20,20)	0	0.053(0.050)	0.058(0.050)	–	0.065(0.050)	0.056(0.050)	–	0.063(0.050)	0.055(0.050)	–	0.060(0.050)
		0.25	0.233(0.229)	0.432(0.399)	–	0.498(0.449)	0.341(0.318)	–	0.396(0.358)	0.259(0.244)	–	0.308(0.278)
		0.50	0.708(0.702)	0.963(0.955)	–	0.976(0.968)	0.915(0.904)	–	0.937(0.923)	0.824(0.812)	–	0.864(0.843)
	(10,10,6,6)	0	0.071(0.050)	0.081(0.050)	–	0.110(0.050)	0.072(0.050)	–	0.101(0.050)	0.065(0.050)	–	0.091(0.050)
		0.25	0.152(0.101)	0.172(0.111)	–	0.268(0.148)	0.160(0.120)	–	0.234(0.130)	0.134(0.100)	–	0.198(0.121)
		0.50	0.450(0.359)	0.616(0.489)	–	0.731(0.550)	0.516(0.438)	–	0.630(0.456)	0.414(0.353)	–	0.534(0.393)
	(20,20,12,12)	0	0.053(0.050)	0.066(0.050)	–	0.077(0.050)	0.063(0.050)	–	0.070(0.050)	0.058(0.050)	–	0.064(0.050)
		0.25	0.233(0.229)	0.319(0.262)	–	0.398(0.319)	0.263(0.225)	–	0.329(0.268)	0.220(0.192)	–	0.280(0.233)
		0.50	0.708(0.702)	0.906(0.864)	–	0.937(0.904)	0.824(0.793)	–	0.872(0.829)	0.722(0.691)	–	0.778(0.737)

The size of tests is $\alpha = 0.05$. The parentheses are the empirically corrected powers for each method.

Table B6. *Under lognormal distribution, approximated size and power of three RSS-based testing methods (Pivot, EL^{B}_{RSS} and EL_{RSS}) and SRS-based testing method (EL_{SRS}).*

				$\rho = 1$			$\rho = 0.9$			$\rho = 0.7$		
H	n_h	δ	EL_{SRS}	*Pivot*	EL^{B}_{RSS}	EL_{RSS}	*Pivot*	EL^{B}_{RSS}	EL_{RSS}	*Pivot*	EL^{B}_{RSS}	EL_{RSS}
2	(8,8)	0	0.130(0.050)	0.120(0.050)	0.148(0.050)	0.139(0.050)	0.113(0.050)	0.147(0.050)	0.131(0.050)	0.114(0.050)	0.151(0.050)	0.131(0.050)
		0.25	0.259(0.051)	0.176(0.034)	0.372(0.074)	0.351(0.117)	0.178(0.038)	0.352(0.069)	0.322(0.109)	0.159(0.026)	0.332(0.046)	0.296(0.096)
		0.50	0.816(0.467)	0.765(0.357)	0.929(0.642)	0.921(0.719)	0.718(0.342)	0.905(0.573)	0.889(0.656)	0.673(0.263)	0.884(0.457)	0.862(0.607)
	(16,16)	0	0.101(0.050)	0.088(0.050)	0.106(0.050)	0.094(0.050)	0.087(0.050)	0.106(0.050)	0.087(0.050)	0.086(0.050)	0.107(0.050)	0.085(0.050)
		0.25	0.415(0.131)	0.350(0.181)	0.543(0.335)	0.527(0.387)	0.325(0.173)	0.507(0.259)	0.483(0.361)	0.288(0.166)	0.479(0.213)	0.448(0.347)
		0.50	0.979(0.873)	0.983(0.930)	0.997(0.987)	0.997(0.991)	0.969(0.903)	0.994(0.966)	0.993(0.980)	0.954(0.881)	0.988(0.939)	0.987(0.976)
4	(8,8,8,8)	0	0.092(0.050)	0.093(0.050)	0.108(0.050)	0.113(0.050)	0.082(0.050)	0.101(0.050)	0.102(0.050)	0.087(0.050)	0.111(0.050)	0.103(0.050)
		0.25	0.414(0.180)	0.514(0.302)	0.691(0.484)	0.706(0.495)	0.430(0.267)	0.606(0.432)	0.588(0.414)	0.349(0.199)	0.538(0.308)	0.512(0.334)
		0.50	0.979(0.915)	0.998(0.981)	1.000(0.999)	1.000(0.999)	0.990(0.960)	0.999(0.994)	0.999(0.993)	0.969(0.909)	0.994(0.971)	0.994(0.975)
	(16,16,16,16)	0	0.078(0.050)	0.069(0.050)	0.071(0.050)	0.072(0.050)	0.068(0.050)	0.076(0.050)	0.071(0.050)	0.070(0.050)	0.085(0.050)	0.075(0.050)
		0.25	0.675(0.551)	0.860(0.790)	0.927(0.889)	0.932(0.903)	0.759(0.685)	0.852(0.785)	0.842(0.785)	0.656(0.560)	0.782(0.676)	0.767(0.690)
		0.50	1.000(0.999)	1.000(1.000)	1.000(1.000)	1.000(1.000)	1.000(1.000)	1.000(1.000)	1.000(1.000)	1.000(0.999)	1.000(1.000)	1.000(1.000)
2	(6,10)	0	0.130(0.050)	0.092(0.050)	–	0.110(0.050)	0.102(0.050)	–	0.120(0.050)	0.105(0.050)	–	0.123(0.050)
		0.25	0.259(0.051)	0.187(0.068)	–	0.359(0.181)	0.181(0.049)	–	0.325(0.141)	0.157(0.039)	–	0.299(0.121)
		0.50	0.816(0.467)	0.819(0.567)	–	0.946(0.852)	0.762(0.466)	–	0.902(0.752)	0.694(0.364)	–	0.869(0.681)
	(12,20)	0	0.101(0.050)	0.078(0.050)	–	0.076(0.050)	0.069(0.050)	–	0.075(0.050)	0.073(0.050)	–	0.077(0.050)
		0.25	0.415(0.131)	0.403(0.267)	–	0.579(0.462)	0.357(0.254)	–	0.517(0.432)	0.314(0.205)	–	0.472(0.378)
		0.50	0.979(0.873)	0.992(0.970)	–	0.999(0.997)	0.983(0.961)	–	0.995(0.991)	0.968(0.933)	–	0.991(0.982)
	(10,6)	0	0.130(0.050)	0.124(0.050)	–	0.153(0.050)	0.132(0.050)	–	0.164(0.050)	0.127(0.050)	–	0.152(0.050)
		0.25	0.259(0.051)	0.144(0.029)	–	0.313(0.086)	0.146(0.021)	–	0.288(0.080)	0.130(0.016)	–	0.274(0.068)
		0.50	0.816(0.467)	0.668(0.258)	–	0.894(0.550)	0.609(0.186)	–	0.833(0.455)	0.569(0.144)	–	0.815(0.437)
	(20,12)	0	0.101(0.050)	0.102(0.050)	–	0.109(0.050)	0.097(0.050)	–	0.104(0.050)	0.096(0.050)	–	0.101(0.050)
		0.25	0.415(0.131)	0.271(0.088)	–	0.472(0.271)	0.264(0.101)	–	0.423(0.269)	0.224(0.079)	-	0.393(0.236)
		0.50	0.979(0.873)	0.953(0.777)	–	0.993(0.970)	0.925(0.755)	–	0.979(0.948)	0.903(0.703)	–	0.973(0.934)

(*Continues*)

Table B6 *(Continued)*

				$\rho = 1$			$\rho = 0.9$			$\rho = 0.7$		
H	n_h	δ	EL_{SRS}	*Pivot*	EL^{B}_{RSS}	EL_{RSS}	*Pivot*	EL^{B}_{RSS}	EL_{RSS}	*Pivot*	EL^{B}_{RSS}	EL_{RSS}
4	(6,6,10,10)	0	0.092(0.050)	0.072(0.050)	–	0.093(0.050)	0.075(0.050)	–	0.091(0.050)	0.071(0.050)	–	0.091(0.050)
		0.25	0.414(0.180)	0.593(0.471)	–	0.754(0.640)	0.478(0.342)	–	0.622(0.484)	0.375(0.277)	–	0.540(0.409)
		0.50	0.979(0.915)	0.999(0.997)	–	1.000(1.000)	0.994(0.984)	–	0.998(0.995)	0.980(0.962)	–	0.993(0.987)
	(12,12,20,20)	0	0.078(0.050)	0.071(0.050)	–	0.075(0.050)	0.062(0.050)	–	0.065(0.050)	0.063(0.050)	–	0.071(0.050)
		0.25	0.675(0.551)	0.915(0.856)	–	0.958(0.937)	0.818(0.780)	–	0.885(0.860)	0.716(0.653)	–	0.815(0.757)
		0.50	1.000(0.999)	1.000(1.000)	–	1.000(1.000)	1.000(1.000)	–	1.000(1.000)	1.000(0.999)	–	1.000(1.000)
	(10,10,6,6)	0	0.092(0.050)	0.100(0.050)	–	0.130(0.050)	0.102(0.050)	–	0.134(0.050)	0.100(0.050)	–	0.125(0.050)
		0.25	0.414(0.180)	0.413(0.177)	–	0.632(0.372)	0.349(0.149)	–	0.528(0.276)	0.290(0.132)	–	0.466(0.263)
		0.50	0.979(0.915)	0.991(0.918)	–	1.000(0.997)	0.968(0.836)	–	0.996(0.965)	0.935(0.786)	–	0.986(0.950)
	(20,20,12,12)	0	0.078(0.050)	0.091(0.050)	–	0.098(0.050)	0.075(0.050)	–	0.087(0.050)	0.070(0.050)	–	0.082(0.050)
		0.25	0.675(0.551)	0.757(0.577)	–	0.875(0.785)	0.647(0.541)	–	0.770(0.689)	0.543(0.435)	–	0.691(0.602)
		0.50	1.000(0.999)	1.000(1.000)	–	1.000(1.000)	0.999(0.998)	–	1.000(1.000)	0.997(0.995)	–	1.000(0.999)

The size of tests is $\alpha = 0.05$. The parentheses are the empirically corrected powers for each method.

Appendix C: Results of bodyfat data application

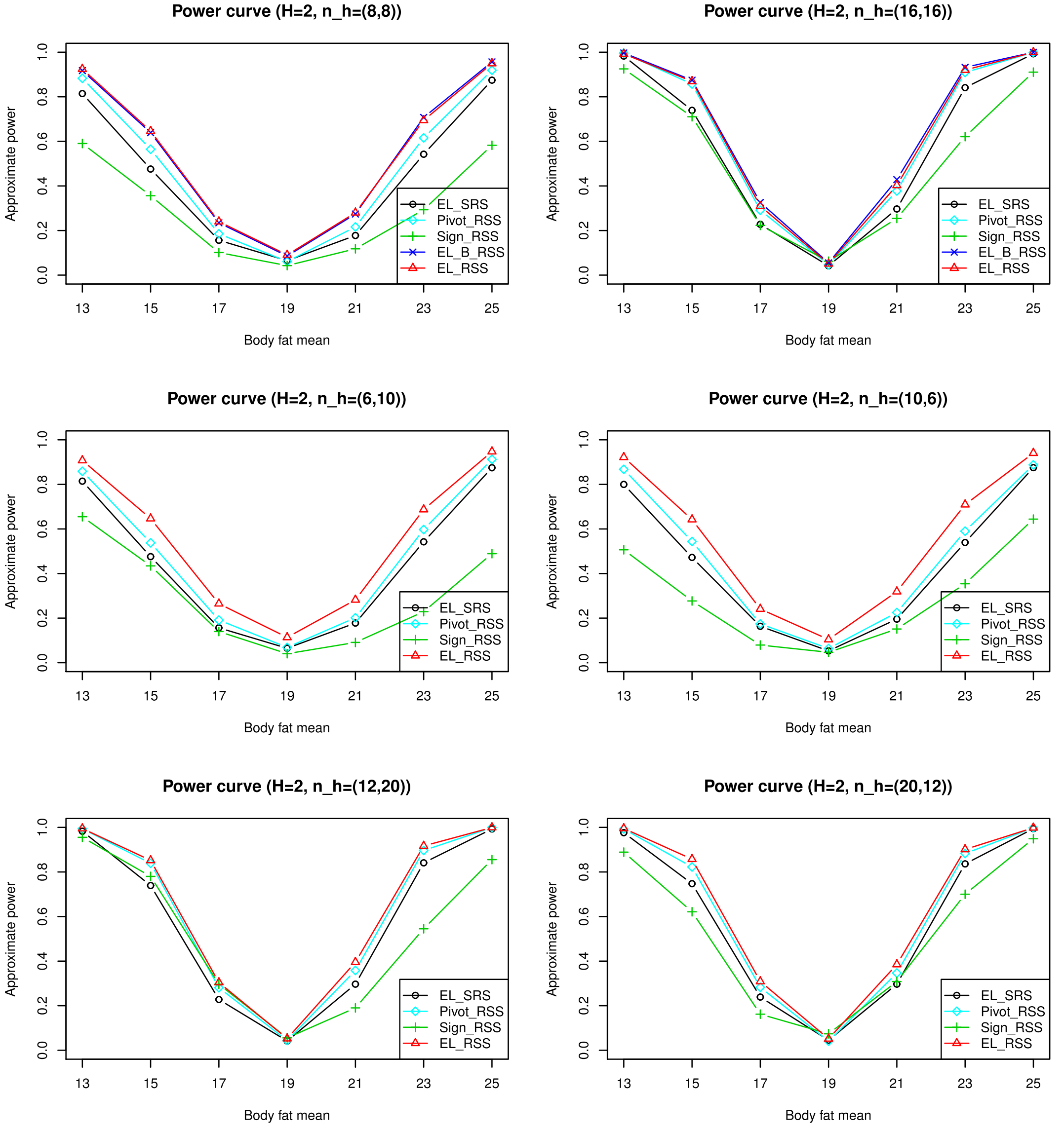

Figure C1. *Approximate power comparison of the four RSS-based testing methods (Pivot, S^{+}_{RSS}, EL^{B}_{RSS} and EL_{RSS}) and the SRS-based testing method (EL_{SRS}) under imperfect ranking by abdomen circumference variable ($\rho = 0.81$) with $H = 2$.*

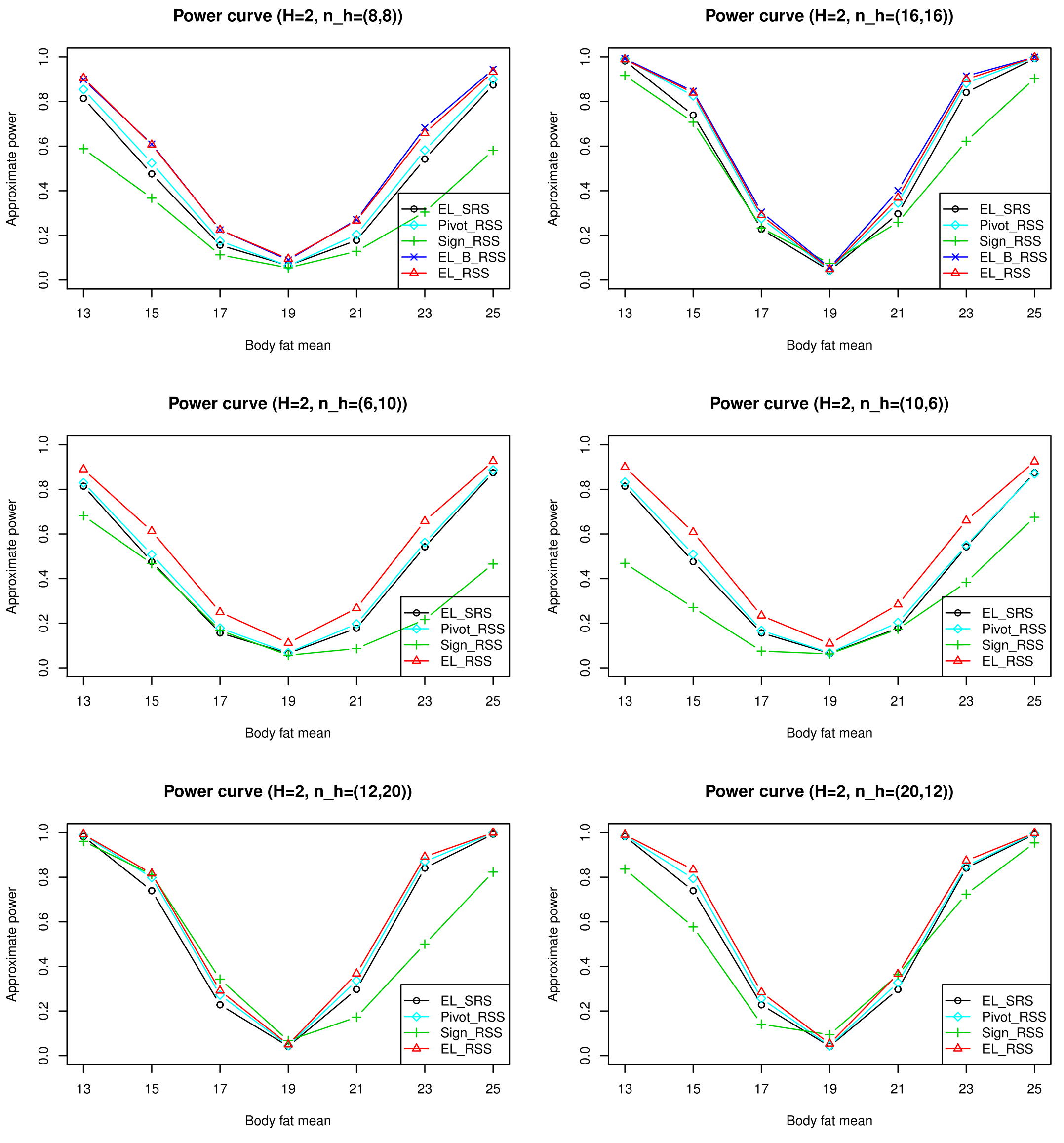

Figure C2. *Approximate power comparison of the four RSS-based testing methods (Pivot,* S^{+}_{RSS}*,* EL^{B}_{RSS} *and* EL_{RSS}*) and the SRS-based testing method (*EL_{SRS}*) under imperfect ranking by chest circumference variable (*$\rho = 0.7$*) with* $H = 2$*.*

Appendix D: Results of JEL extension

Table D1. *Under normal distribution, approximated size and power of two RSS JEL methods* (JEL^{Z}_{RSS} *and* EL_{RSS}).

			$\rho = 1$		$\rho = 0.9$		$\rho = 0.7$	
H	n_h	δ	JEL^{Z}_{RSS}	EL_{RSS}	JEL^{Z}_{RSS}	EL_{RSS}	JEL^{Z}_{RSS}	EL_{RSS}
2	(8,8)	0	0.062(0.050)	0.084(0.050)	0.068(0.050)	0.090(0.050)	0.062(0.050)	0.087(0.050)
		0.25	0.242(0.212)	0.286(0.212)	0.234(0.181)	0.276(0.184)	0.211(0.186)	0.253(0.185)
		0.5	0.664(0.622)	0.715(0.623)	0.638(0.569)	0.694(0.569)	0.584(0.547)	0.640(0.549)
	(16,16)	0	0.057(0.050)	0.066(0.050)	0.053(0.050)	0.066(0.050)	0.059(0.050)	0.068(0.050)
		0.25	0.411(0.384)	0.436(0.385)	0.378(0.370)	0.410(0.369)	0.339(0.310)	0.361(0.312)
		0.5	0.918(0.904)	0.926(0.904)	0.905(0.901)	0.917(0.900)	0.86(0.842)	0.875(0.841)
4	(8,8,8,8)	0	0.050(0.050)	0.083(0.050)	0.049(0.050)	0.075(0.050)	0.051(0.050)	0.076(0.050)
		0.25	0.548(0.547)	0.633(0.546)	0.460(0.469)	0.542(0.465)	0.353(0.351)	0.442(0.351)
		0.5	0.983(0.983)	0.992(0.982)	0.960(0.962)	0.979(0.961)	0.895(0.894)	0.931(0.895)
	(16,16,16,16)	0	0.046(0.050)	0.059(0.050)	0.052(0.050)	0.063(0.050)	0.046(0.050)	0.058(0.050)
		0.25	0.841(0.853)	0.866(0.853)	0.756(0.751)	0.783(0.753)	0.635(0.646)	0.670(0.648)
		0.5	1.000(1.000)	1.000(1.000)	1.000(1.000)	1.000(1.000)	0.997(0.997)	0.998(0.997)
2	(6,10)	0	0.071(0.050)	0.098(0.050)	0.067(0.050)	0.097(0.050)	0.075(0.050)	0.102(0.050)
		0.25	0.240(0.196)	0.304(0.188)	0.218(0.181)	0.281(0.177)	0.196(0.148)	0.263(0.148)
		0.5	0.619(0.553)	0.699(0.554)	0.601(0.548)	0.682(0.541)	0.541(0.467)	0.624(0.470)
	(12,20)	0	0.054(0.050)	0.068(0.050)	0.063(0.050)	0.075(0.050)	0.057(0.050)	0.069(0.050)
		0.25	0.371(0.354)	0.407(0.354)	0.348(0.306)	0.391(0.305)	0.299(0.281)	0.332(0.285)
		0.5	0.890(0.880)	0.912(0.882)	0.877(0.844)	0.897(0.848)	0.821(0.801)	0.850(0.809)
	(10,6)	0	0.064(0.050)	0.096(0.050)	0.072(0.050)	0.104(0.050)	0.068(0.050)	0.102(0.050)
		0.25	0.223(0.186)	0.283(0.189)	0.225(0.172)	0.290(0.178)	0.206(0.158)	0.269(0.171)
		0.5	0.626(0.574)	0.713(0.586)	0.578(0.495)	0.663(0.512)	0.535(0.461)	0.622(0.483)
	(20,12)	0	0.056(0.050)	0.070(0.050)	0.054(0.050)	0.068(0.050)	0.060(0.050)	0.074(0.050)
		0.25	0.367(0.340)	0.406(0.344)	0.359(0.348)	0.394(0.349)	0.306(0.275)	0.337(0.277)
		0.5	0.902(0.886)	0.919(0.889)	0.869(0.861)	0.890(0.862)	0.833(0.808)	0.856(0.811)
4	(6,6,10,10)	0	0.049(0.050)	0.085(0.050)	0.052(0.050)	0.087(0.050)	0.050(0.050)	0.089(0.050)
		0.25	0.505(0.511)	0.601(0.519)	0.414(0.406)	0.520(0.414)	0.326(0.326)	0.423(0.324)
		0.5	0.974(0.974)	0.987(0.978)	0.944(0.940)	0.971(0.942)	0.868(0.867)	0.918(0.865)
	(12,12,20,20)	0	0.048(0.050)	0.065(0.050)	0.050(0.050)	0.066(0.050)	0.045(0.050)	0.063(0.050)
		0.25	0.822(0.829)	0.855(0.830)	0.717(0.718)	0.766(0.717)	0.603(0.624)	0.660(0.622)
		0.5	1.000(1.000)	1.000(1.000)	0.999(0.999)	1.000(0.999)	0.993(0.993)	0.994(0.993)

(*Continues*)

Table D1 *(Continued)*

H	n_h	δ	$\rho = 1$		$\rho = 0.9$		$\rho = 0.7$	
			$\text{JEL}^{\text{Z}}_{\text{RSS}}$	EL_{RSS}	$\text{JEL}^{\text{Z}}_{\text{RSS}}$	EL_{RSS}	$\text{JEL}^{\text{Z}}_{\text{RSS}}$	EL_{RSS}
	(10,10,6,6)	0	0.057(0.050)	0.095(0.050)	0.052(0.050)	0.085(0.050)	0.048(0.050)	0.077(0.050)
		0.25	0.510(0.481)	0.621(0.490)	0.430(0.423)	0.541(0.428)	0.335(0.349)	0.440(0.348)
		0.5	0.978(0.973)	0.992(0.975)	0.942(0.939)	0.968(0.942)	0.863(0.873)	0.921(0.874)
	(20,20,12,12)	0	0.050(0.050)	0.066(0.050)	0.049(0.050)	0.066(0.050)	0.051(0.050)	0.065(0.050)
		0.25	0.825(0.826)	0.857(0.829)	0.718(0.722)	0.761(0.724)	0.584(0.581)	0.634(0.583)
		0.5	1.000(1.000)	1.000(1.000)	0.999(0.999)	1.000(1.000)	0.994(0.994)	0.996(0.994)

The size of tests is $\alpha = 0.05$. The parentheses are the empirically corrected powers for each method.

Table D2. *Under t distribution, approximated size and power of two RSS JEL methods* (JEL^{Z}_{RSS} *and* EL_{RSS}).

			$\rho = 1$		$\rho = 0.9$		$\rho = 0.7$	
H	n_h	δ	JEL^{Z}_{RSS}	EL_{RSS}	JEL^{Z}_{RSS}	EL_{RSS}	JEL^{Z}_{RSS}	EL_{RSS}
2	(8,8)	0	0.075(0.050)	0.102(0.050)	0.073(0.050)	0.097(0.050)	0.074(0.050)	0.101(0.050)
		0.25	0.201(0.151)	0.242(0.152)	0.196(0.150)	0.239(0.145)	0.184(0.137)	0.229(0.137)
		0.5	0.511(0.444)	0.576(0.441)	0.465(0.397)	0.525(0.394)	0.430(0.358)	0.490(0.360)
	(16,16)	0	0.060(0.050)	0.070(0.050)	0.066(0.050)	0.077(0.050)	0.065(0.050)	0.073(0.050)
		0.25	0.290(0.257)	0.308(0.251)	0.261(0.222)	0.281(0.221)	0.243(0.205)	0.265(0.200)
		0.5	0.734(0.704)	0.743(0.688)	0.706(0.662)	0.720(0.651)	0.661(0.618)	0.676(0.601)
4	(8,8,8,8)	0	0.056(0.050)	0.087(0.050)	0.052(0.050)	0.089(0.050)	0.055(0.050)	0.085(0.050)
		0.25	0.374(0.357)	0.448(0.345)	0.309(0.305)	0.385(0.295)	0.251(0.242)	0.318(0.240)
		0.5	0.864(0.852)	0.894(0.835)	0.785(0.782)	0.836(0.772)	0.703(0.694)	0.769(0.688)
	(16,16,16,16)	0	0.054(0.050)	0.069(0.050)	0.052(0.050)	0.063(0.050)	0.050(0.050)	0.064(0.050)
		0.25	0.611(0.604)	0.637(0.588)	0.531(0.525)	0.565(0.516)	0.429(0.427)	0.460(0.423)
		0.5	0.985(0.985)	0.978(0.971)	0.970(0.970)	0.968(0.958)	0.927(0.926)	0.930(0.917)
2	(6,10)	0	0.078(0.050)	0.118(0.050)	0.073(0.050)	0.106(0.050)	0.073(0.050)	0.108(0.050)
		0.25	0.205(0.157)	0.255(0.144)	0.198(0.152)	0.251(0.149)	0.183(0.145)	0.238(0.138)
		0.5	0.499(0.417)	0.579(0.410)	0.464(0.401)	0.545(0.399)	0.430(0.372)	0.513(0.371)
	(12,20)	0	0.065(0.050)	0.082(0.050)	0.063(0.050)	0.078(0.050)	0.057(0.050)	0.073(0.050)
		0.25	0.295(0.259)	0.315(0.251)	0.264(0.238)	0.290(0.228)	0.243(0.225)	0.268(0.213)
		0.5	0.706(0.679)	0.715(0.654)	0.693(0.662)	0.709(0.634)	0.643(0.625)	0.669(0.600)
	(10,6)	0	0.076(0.050)	0.116(0.050)	0.073(0.050)	0.110(0.050)	0.074(0.050)	0.110(0.050)
		0.25	0.170(0.129)	0.234(0.136)	0.168(0.126)	0.225(0.136)	0.159(0.121)	0.211(0.127)
		0.5	0.474(0.401)	0.558(0.401)	0.439(0.363)	0.518(0.388)	0.400(0.335)	0.487(0.334)
	(20,12)	0	0.067(0.050)	0.079(0.050)	0.060(0.050)	0.073(0.050)	0.058(0.050)	0.073(0.050)
		0.25	0.255(0.217)	0.290(0.217)	0.235(0.206)	0.264(0.206)	0.217(0.197)	0.248(0.198)
		0.5	0.721(0.678)	0.746(0.665)	0.680(0.649)	0.706(0.635)	0.641(0.612)	0.667(0.601)
4	(6,6,10,10)	0	0.056(0.050)	0.095(0.050)	0.054(0.050)	0.094(0.050)	0.050(0.050)	0.091(0.050)
		0.25	0.369(0.361)	0.451(0.346)	0.303(0.295)	0.383(0.284)	0.248(0.248)	0.327(0.243)
		0.5	0.832(0.828)	0.876(0.809)	0.763(0.756)	0.826(0.742)	0.661(0.661)	0.744(0.652)
	(12,12,20,20)	0	0.053(0.050)	0.069(0.050)	0.053(0.050)	0.070(0.050)	0.050(0.050)	0.068(0.050)
		0.25	0.582(0.575)	0.607(0.550)	0.498(0.490)	0.530(0.466)	0.420(0.420)	0.466(0.408)
		0.5	0.974(0.972)	0.960(0.950)	0.956(0.955)	0.950(0.931)	0.915(0.915)	0.916(0.896)
	(10,10,6,6)	0	0.055(0.050)	0.100(0.050)	0.057(0.050)	0.097(0.050)	0.051(0.050)	0.089(0.050)
		0.25	0.332(0.320)	0.439(0.324)	0.263(0.249)	0.364(0.253)	0.215(0.212)	0.304(0.210)
		0.5	0.854(0.846)	0.896(0.824)	0.772(0.758)	0.839(0.745)	0.664(0.659)	0.748(0.646)
	(20,20,12,12)	0	0.048(0.050)	0.068(0.050)	0.052(0.050)	0.070(0.050)	0.051(0.050)	0.066(0.050)
		0.25	0.579(0.591)	0.627(0.581)	0.497(0.488)	0.543(0.484)	0.393(0.391)	0.444(0.387)
		0.5	0.988(0.988)	0.982(0.977)	0.970(0.969)	0.974(0.961)	0.933(0.932)	0.943(0.927)

The size of tests is $\alpha = 0.05$. The parentheses are the empirically corrected powers for each method.

Table D3. *Under gamma distribution, approximated size and power of two RSS JEL methods* (JEL^{Z}_{RSS} *and* EL_{RSS}).

			$\rho = 1$		$\rho = 0.9$		$\rho = 0.7$	
H	n_h	δ	JEL^{Z}_{RSS}	EL_{RSS}	JEL^{Z}_{RSS}	EL_{RSS}	JEL^{Z}_{RSS}	EL_{RSS}
2	(8,8)	0	0.082(0.050)	0.114(0.050)	0.082(0.050)	0.111(0.050)	0.078(0.050)	0.110(0.050)
		0.25	0.120(0.066)	0.165(0.062)	0.102(0.057)	0.146(0.055)	0.101(0.058)	0.140(0.051)
		0.5	0.291(0.186)	0.367(0.176)	0.277(0.189)	0.347(0.178)	0.256(0.169)	0.316(0.147)
	(16,16)	0	0.065(0.050)	0.073(0.050)	0.071(0.050)	0.080(0.050)	0.065(0.050)	0.073(0.050)
		0.25	0.170(0.133)	0.197(0.145)	0.146(0.109)	0.170(0.116)	0.139(0.112)	0.159(0.118)
		0.5	0.548(0.491)	0.584(0.508)	0.507(0.416)	0.542(0.437)	0.465(0.407)	0.499(0.423)
4	(8,8,8,8)	0	0.054(0.050)	0.083(0.050)	0.055(0.050)	0.083(0.050)	0.056(0.050)	0.086(0.050)
		0.25	0.203(0.189)	0.289(0.201)	0.177(0.165)	0.246(0.176)	0.137(0.133)	0.209(0.134)
		0.5	0.697(0.676)	0.792(0.695)	0.583(0.569)	0.683(0.581)	0.475(0.456)	0.578(0.465)
	(16,16,16,16)	0	0.051(0.050)	0.067(0.050)	0.052(0.050)	0.063(0.050)	0.049(0.050)	0.065(0.050)
		0.25	0.414(0.408)	0.463(0.421)	0.347(0.340)	0.396(0.347)	0.274(0.275)	0.322(0.279)
		0.5	0.951(0.950)	0.962(0.952)	0.894(0.891)	0.913(0.894)	0.804(0.805)	0.833(0.808)
2	(6,10)	0	0.070(0.050)	0.100(0.050)	0.069(0.050)	0.098(0.050)	0.070(0.050)	0.098(0.050)
		0.25	0.122(0.089)	0.158(0.075)	0.110(0.077)	0.151(0.066)	0.102(0.065)	0.139(0.061)
		0.5	0.319(0.253)	0.387(0.219)	0.296(0.225)	0.352(0.203)	0.264(0.193)	0.325(0.181)
	(12,20)	0	0.063(0.050)	0.072(0.050)	0.065(0.050)	0.072(0.050)	0.063(0.050)	0.069(0.050)
		0.25	0.175(0.149)	0.200(0.155)	0.174(0.135)	0.197(0.141)	0.159(0.125)	0.179(0.131)
		0.5	0.576(0.535)	0.603(0.547)	0.514(0.453)	0.540(0.463)	0.452(0.407)	0.482(0.421)
	(10,6)	0	0.083(0.050)	0.146(0.054)	0.091(0.050)	0.144(0.050)	0.091(0.050)	0.142(0.050)
		0.25	0.104(0.060)	0.173(0.077)	0.092(0.037)	0.165(0.061)	0.090(0.037)	0.157(0.050)
		0.5	0.226(0.130)	0.353(0.159)	0.218(0.108)	0.325(0.145)	0.194(0.103)	0.296(0.114)
	(20,12)	0	0.069(0.050)	0.085(0.050)	0.068(0.050)	0.087(0.050)	0.065(0.050)	0.079(0.050)
		0.25	0.142(0.091)	0.182(0.123)	0.125(0.091)	0.166(0.109)	0.115(0.087)	0.152(0.104)
		0.5	0.435(0.330)	0.513(0.402)	0.413(0.335)	0.487(0.382)	0.382(0.316)	0.453(0.360)
4	(6,6,10,10)	0	0.051(0.050)	0.076(0.050)	0.053(0.050)	0.082(0.050)	0.050(0.050)	0.076(0.050)
		0.25	0.226(0.226)	0.305(0.234)	0.182(0.175)	0.248(0.180)	0.144(0.144)	0.205(0.150)
		0.5	0.735(0.734)	0.817(0.744)	0.625(0.617)	0.717(0.624)	0.502(0.501)	0.599(0.513)
	(12,12,20,20)	0	0.045(0.050)	0.058(0.050)	0.048(0.050)	0.060(0.050)	0.049(0.050)	0.062(0.050)
		0.25	0.448(0.466)	0.492(0.469)	0.360(0.365)	0.402(0.369)	0.272(0.277)	0.310(0.276)
		0.5	0.965(0.969)	0.974(0.969)	0.914(0.916)	0.933(0.918)	0.824(0.827)	0.850(0.827)
	(10,10,6,6)	0	0.061(0.050)	0.108(0.050)	0.056(0.050)	0.107(0.050)	0.059(0.050)	0.108(0.050)
		0.25	0.157(0.131)	0.281(0.159)	0.127(0.112)	0.234(0.133)	0.108(0.090)	0.192(0.096)
		0.5	0.584(0.532)	0.742(0.571)	0.475(0.444)	0.641(0.473)	0.369(0.331)	0.536(0.349)
	(20,20,12,12)	0	0.052(0.050)	0.072(0.050)	0.046(0.050)	0.067(0.050)	0.045(0.050)	0.061(0.050)
		0.25	0.325(0.318)	0.400(0.334)	0.264(0.277)	0.336(0.285)	0.221(0.244)	0.279(0.254)
		0.5	0.917(0.914)	0.947(0.920)	0.814(0.825)	0.866(0.833)	0.706(0.732)	0.771(0.740)

The size of tests is $\alpha = 0.05$. The parentheses are the empirically corrected powers for each method.

Table D4. *Under lognormal distribution, approximated size and power of two RSS JEL methods* (JEL^{Z}_{RSS} *and* EL_{RSS}).

			$\rho = 1$		$\rho = 0.9$		$\rho = 0.7$	
H	n_h	δ	JEL^{Z}_{RSS}	EL_{RSS}	JEL^{Z}_{RSS}	EL_{RSS}	JEL^{Z}_{RSS}	EL_{RSS}
2	(8,8)	0	0.111(0.050)	0.139(0.050)	0.108(0.050)	0.131(0.050)	0.107(0.050)	0.131(0.050)
		0.25	0.244(0.057)	0.334(0.115)	0.225(0.054)	0.320(0.106)	0.220(0.049)	0.306(0.091)
		0.5	0.881(0.545)	0.928(0.720)	0.822(0.475)	0.883(0.651)	0.797(0.442)	0.857(0.606)
	(16,16)	0	0.094(0.050)	0.096(0.050)	0.091(0.050)	0.090(0.050)	0.082(0.050)	0.084(0.050)
		0.25	0.476(0.211)	0.529(0.380)	0.438(0.207)	0.492(0.365)	0.420(0.232)	0.465(0.348)
		0.5	0.994(0.970)	0.996(0.990)	0.985(0.941)	0.990(0.977)	0.982(0.941)	0.985(0.969)
4	(8,8,8,8)	0	0.080(0.050)	0.113(0.050)	0.070(0.050)	0.102(0.050)	0.074(0.050)	0.103(0.050)
		0.25	0.559(0.328)	0.708(0.500)	0.455(0.336)	0.599(0.415)	0.379(0.265)	0.516(0.332)
		0.5	1.000(0.997)	1.000(1.000)	0.993(0.987)	0.998(0.992)	0.980(0.961)	0.991(0.973)
	(16,16,16,16)	0	0.067(0.050)	0.075(0.050)	0.060(0.050)	0.071(0.050)	0.065(0.050)	0.076(0.050)
		0.25	0.907(0.853)	0.932(0.903)	0.813(0.767)	0.850(0.800)	0.727(0.675)	0.774(0.716)
		0.5	1.000(1.000)	1.000(1.000)	1.000(1.000)	1.000(1.000)	1.000(0.999)	1.000(1.000)
2	(6,10)	0	0.098(0.050)	0.116(0.050)	0.094(0.050)	0.113(0.050)	0.095(0.050)	0.115(0.050)
		0.25	0.296(0.119)	0.363(0.155)	0.272(0.133)	0.337(0.158)	0.239(0.107)	0.306(0.138)
		0.5	0.917(0.779)	0.943(0.817)	0.881(0.735)	0.910(0.778)	0.842(0.673)	0.885(0.736)
	(12,20)	0	0.081(0.050)	0.080(0.050)	0.079(0.050)	0.084(0.050)	0.081(0.050)	0.083(0.050)
		0.25	0.549(0.384)	0.580(0.487)	0.494(0.355)	0.524(0.422)	0.456(0.327)	0.492(0.388)
		0.5	0.997(0.993)	0.998(0.996)	0.992(0.981)	0.993(0.988)	0.987(0.971)	0.99(0.981)
	(10,6)	0	0.122(0.050)	0.170(0.050)	0.119(0.050)	0.165(0.050)	0.116(0.050)	0.160(0.050)
		0.25	0.165(0.024)	0.321(0.086)	0.143(0.023)	0.286(0.073)	0.138(0.020)	0.273(0.062)
		0.5	0.750(0.213)	0.888(0.528)	0.671(0.184)	0.841(0.461)	0.650(0.170)	0.810(0.440)
	(20,12)	0	0.100(0.050)	0.103(0.050)	0.096(0.050)	0.098(0.050)	0.094(0.050)	0.096(0.050)
		0.25	0.358(0.006)	0.465(0.294)	0.321(0.034)	0.421(0.260)	0.301(0.024)	0.396(0.247)
		0.5	0.988(0.193)	0.994(0.977)	0.968(0.605)	0.981(0.948)	0.957(0.541)	0.971(0.938)
4	(6,6,10,10)	0	0.070(0.050)	0.097(0.050)	0.067(0.050)	0.096(0.050)	0.067(0.050)	0.097(0.050)
		0.25	0.658(0.535)	0.754(0.611)	0.533(0.473)	0.633(0.509)	0.437(0.355)	0.541(0.403)
		0.5	1.000(1.000)	1.000(1.000)	0.996(0.994)	0.999(0.995)	0.989(0.982)	0.994(0.987)
	(12,12,20,20)	0	0.066(0.050)	0.072(0.050)	0.056(0.050)	0.066(0.050)	0.055(0.050)	0.064(0.050)
		0.25	0.942(0.918)	0.953(0.931)	0.868(0.855)	0.890(0.870)	0.777(0.758)	0.811(0.780)
		0.5	1.000(1.000)	1.000(1.000)	1.000(1.000)	1.000(1.000)	1.000(0.999)	1.000(1.000)
	(10,10,6,6)	0	0.089(0.050)	0.133(0.050)	0.074(0.050)	0.124(0.050)	0.076(0.050)	0.113(0.050)

(*Continues*)

Table D4 *(Continued)*

			$\rho = 1$		$\rho = 0.9$		$\rho = 0.7$	
H	n_h	δ	JEL^{Z}_{RSS}	EL_{RSS}	JEL^{Z}_{RSS}	EL_{RSS}	JEL^{Z}_{RSS}	EL_{RSS}
		0.25	0.391(0.069)	0.641(0.363)	0.316(0.165)	0.523(0.314)	0.257(0.096)	0.458(0.254)
		0.5	0.997(0.720)	1.000(0.997)	0.970(0.893)	0.994(0.973)	0.942(0.758)	0.981(0.939)
	(20,20,12,12)	0	0.078(0.050)	0.096(0.050)	0.070(0.050)	0.086(0.050)	0.065(0.050)	0.079(0.050)
		0.25	0.798(0.556)	0.870(0.790)	0.691(0.559)	0.778(0.678)	0.612(0.497)	0.704(0.623)
		0.5	1.000(1.000)	1.000(1.000)	1.000(1.000)	1.000(1.000)	1.000(0.999)	1.000(1.000)

The size of tests is $\alpha = 0.05$. The parentheses are the empirically corrected powers for each method.

[*Received July 2023; accepted June 2024*]