



The optimal proper connection number of a graph with given independence number

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ABSTRACT

An edge-colored connected graph G is *properly connected* if between every pair of distinct vertices, there exists a path such that no two adjacent edges have the same color. Fujita (2019) introduced the *optimal proper connection number* $pc_{\text{opt}}(G)$ for a monochromatic connected graph G , to make a connected graph properly connected efficiently. More precisely, $pc_{\text{opt}}(G)$ is the smallest integer $p + q$ when one converts a given monochromatic graph G into a properly connected graph by recoloring p edges with q colors.

In this paper, we show that $pc_{\text{opt}}(G)$ has an upper bound in terms of the independence number $\alpha(G)$. Namely, we prove that for a connected graph G , $pc_{\text{opt}}(G) \leq \frac{5\alpha(G)-1}{2}$. Moreover, for the case $\alpha(G) \leq 3$, we improve the upper bound to 4, which is tight.

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1. Introduction

In this paper, we consider simple and finite graphs. We refer the reader to [1] for terminology and notation not defined here. For a graph G , let $\alpha(G)$ be the independence number of G and $\nu(G)$ be the size of a maximum matching of G . An edge-colored graph is *properly colored* if no two adjacent edges have a same color. The notion on properly colored graphs sometimes plays an important role in a variety of fields related to the application of graphs. For example, properly colored paths and cycles appear in the context of some problems in genetics [2,3] and social sciences [4]. An edge-colored connected graph G is *properly connected* if between every pair of distinct vertices, there exists a path that is properly colored. As described in [5], making a connected graph G properly connected is important in view of some real applications for building an efficient communication network.

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Motivated by such applications, Borozan et al. [6] introduced a new notion called the *proper connection number* $pc(G)$ of a connected graph G , that is, the minimum number of colors needed to color the edges of G to make it properly connected. This notion attracts much attention on both theoretical and practical aspects in graph theory and a lot of work has been done so far in this area of study (see e.g. [7–12]). Their work mainly explored the minimum number of colors needed to make G properly connected, but most of them do not provide any information on how to make a connected graph properly connected efficiently. Recently, Fujita [13] focused on this point of view and introduced the following notion.

In what follows, for technical reasons, we will firstly assume that a graph G is always connected and all edges of G are colored by color 1 (thus, we say that G is a *monochromatic* graph). We then consider recoloring some edges of G so that G is properly connected. Keeping this requirement in mind, the *optimal proper connection number* of a monochromatic graph G , denoted by $pc_{\text{opt}}(G)$, is defined as follows:

$$pc_{\text{opt}}(G) = \min \{p + q \mid \text{By recoloring } p \text{ edges of } G \text{ with } q \text{ new colors, } G \text{ can be made properly connected}\}.$$

By definition, it is obvious that $\alpha(G) = 1$ if and only if $pc_{\text{opt}}(G) = 0$; moreover, $pc_{\text{opt}}(G) \neq 1$ holds for any graph G .

For integers $m \geq n \geq 2$ with $m + n \geq 9$, Fujita [13] showed that $pc_{\text{opt}}(K_{m,n}) = 4$ for $n = 2, 3$ and $pc_{\text{opt}}(K_{m,n}) = 5$ for $n \geq 4$, and also showed that $pc_{\text{opt}}(T) = n - 2 - \nu(T) + \Delta(T)$ for any tree T of order $n \geq 2$, where $\Delta(T)$ is the maximum degree of T .

For graphs with small independence number, we have the following theorem.

Theorem 1.1 ([13]). *For a connected graph G with $\alpha(G) = 2$, $pc_{\text{opt}}(G) \leq 3$ and the bound is tight.*

In this paper, we generalize an upper bound of this result for graphs with an arbitrary independence number. Indeed, we show that the optimal proper connection number of a graph G has an upper bound in terms of the independence number of G as follows.

Theorem 1.2. *For a connected graph G , $pc_{\text{opt}}(G) \leq \frac{5\alpha(G)-1}{2}$.*

Along this line, we propose the following conjecture.

Conjecture 1. *For a connected graph G with $\alpha(G) \geq 3$, $pc_{\text{opt}}(G) \leq 2\alpha(G) - 2$.*

The bound on $pc_{\text{opt}}(G)$ is best possible if the conjecture is true. To see this, consider a star $K_{1,m}$ and note that $pc_{\text{opt}}(K_{1,m}) = 2m - 2 = 2\alpha(K_{1,m}) - 2$. Our second result is to show that the conjecture is true for connected graphs with independence number three.

Theorem 1.3. *For a connected graph G with $\alpha(G) = 3$, $pc_{\text{opt}}(G) \leq 4$.*

In the proofs of our results, we make use of an extremal structure of graphs with a fixed independence number. Recall that a vertex v of a connected graph G is a *cut-vertex* if $G - v$ is disconnected. A connected graph G is α -*minimal* if for every $v \in V(G)$, either v is a cut-vertex of G or $\alpha(G - v) < \alpha(G)$. We prove that every α -minimal graph G has at most $2\alpha(G) - 1$ vertices (see Lemma 2.1), where the bound on $|V(G)|$ is best possible. To see this, consider the case where G is a path with an odd number of vertices. As we will observe later, this lemma is very useful in our problem, and we believe that the lemma could also be useful for some other extremal problems in graphs with given independence number. As examples of such extremal problems, we briefly introduce some of them: We start with a famous structural result due to Chvátal and

Erdős [14]. Their theorem states that any connected graph G such that $\alpha(G) \leq \kappa(G)$ contains a Hamiltonian cycle, where $\kappa(G)$ is the smallest size of a vertex set S such that $G - S$ is disconnected or K_1 . The bound on $\alpha(G)$ is tight, but motivated by this theorem, O et al. [15] gave the tight upper bound $\frac{\kappa(G)(|V(G)| + \alpha(G) - \kappa(G))}{\alpha(G)}$ on the order of longest cycles in a graph G . Aside from the connectivity of a graph, Egawa et al. [16] asked the maximum order $f(k, a)$ of a graph G with $\alpha(G) \leq a$ and with no k vertex-disjoint cycles and they determined it for the case $1 \leq a \leq 5$ or $1 \leq k \leq 2$. Later, Fujita [17] settled the case $k = 3$ in this extremal problem. Along a slightly different line, Fujita et al. [18] investigated the least value $g(l, a)$ such that any graph G of order at least $g(l, a)$ with $\alpha(G) \leq a$ contains an l -connected subgraph of order at least $\frac{|V(G)|}{a}$ and they determined it for the cases $a = 2, 3$. Although we will not introduce further known results anymore, there are many directions to explore the extremal structure of graphs with given independence number. As we can learn from the above research progress, our standard approach to these extremal problems is to work on the partial problems on graphs with small independence number as a first step. In this way, we might obtain some nice clue to settle the cases (ideally, all the cases) on graphs with a larger fixed independence number. Indeed, Lemma 2.1 can be such a clue to some degree (see Remark 2.4). We present the proof of Theorem 1.3 along the natural approach and it could contribute to provide some nice observation towards some other extremal problems on graphs G with a fixed independence number.

The paper is organized as follows. Section 2 collects key observations for the proofs. In Section 3, we prove Theorem 1.2. We prove Theorem 1.3 in Section 4.

2. Preliminaries

Here are some definitions and notation. We denote $\{1, \dots, n\}$ by $[n]$. Let G be a graph. For $A, B \subset V(G)$, $E_G(A, B)$ is the set of edges joining a vertex in A and a vertex in B . For a vertex $x \in V(G)$, let $N_G(x) = \{y \in V(G) \mid xy \in E(G)\}$ and $N_G[x] = N_G(x) \cup \{x\}$. For $S \subset V(G)$, $G[S]$ and $G - S$ are the subgraph induced by S and $V(G) - S$, respectively. If $S = \{v\}$, then we denote $G - S$ by $G - v$. For an edge uw of G , $G - uw$ means the graph obtained from G by deleting uw . For a subgraph H of G , when $uw \in E(G) \setminus E(H)$ for two vertices u, w of H , $H + uw$ means the graph obtained by adding uw to H . We say a vertex v dominates a set $Z \subset V(G)$ if v is adjacent to every vertex in Z .

For a tree T and $x, y \in V(T)$, let $P_T(x, y)$ denote the path in T from x to y . For two vertex-disjoint paths P and Q of a graph G , if the last vertex of P is adjacent to the first vertex of Q , then $P + Q$ denotes the path along P and Q .

Lemma 2.1. *For an α -minimal connected graph G , $|V(G)| \leq 2\alpha(G) - 1$.*

Proof. We use induction on $\alpha(G)$. If $\alpha(G) = 1$ then G is K_1 and so it is clear. Suppose that the lemma holds for every α -minimal connected graph with independence number less than m for some $m \geq 2$. Let G be an α -minimal connected graph with independence number m . Suppose to the contrary that $|V(G)| \geq 2\alpha(G)$. Let I be a maximum independent set of G . Note that $\alpha(G) = |I| \leq \alpha(G - u)$ for every vertex u in $V(G) \setminus I$, and therefore from the α -minimality, it follows that every vertex in $V(G) \setminus I$ must be a cut-vertex of G . Take a vertex $x \in V(G) \setminus I$ so that the order of a minimum component of $G - x$ is as small as possible. Let G_1, G_2, \dots, G_k ($k \geq 2$) be the components of $G - x$. We may assume that $|V(G_1)| \leq |V(G_2)| \leq \dots \leq |V(G_k)|$. If G_1 contains a vertex $y \in V(G) \setminus I$, then $G - y$ has a smaller component than G_1 . To see this, note that the component of $G - y$ containing x has at least $|V(G) \setminus V(G_1)|$ vertices, meaning that other components of $G - y$ have less than $|V(G_1)|$ vertices, respectively. However, this is a contradiction to the choice of x . Thus $V(G_1) = \{v\}$ for some vertex $v \in I$. Note that since $G - v$ is connected, by the α -minimality, $\alpha(G - v) < \alpha(G)$ and v belongs to every maximum independent set of G .

Claim 2.2. If $|V(G_i)| \geq 2\alpha(G_i)$ for some $i \in \{2, \dots, k\}$, then there is a vertex $v_i \in V(G_i)$ such that $N_G(x) \cap V(G_i) = \{v_i\}$, $G_i - v_i$ is connected, $\alpha(G_i - v_i) = \alpha(G_i)$, and $\alpha(G - v_i) = \alpha(G)$.

Proof. Suppose that there exists $i \in \{2, \dots, k\}$ such that $|V(G_i)| \geq 2\alpha(G_i)$. Then, by the induction hypothesis, G_i is not α -minimal. Hence there exists a vertex $v_i \in V(G_i)$ such that $G_i - v_i$ is connected and $\alpha(G_i - v_i) = \alpha(G_i)$. By taking a maximum independent set I'_i of $G_i - v_i$, $I' = (I - V(G_i)) \cup I'_i$ is a maximum independent set of $G - v_i$, since $|I'| = |(I - V(G_i)) \cup I'_i| = \sum_{i \in [k]} \alpha(G_i) = \alpha(G)$. Thus $\alpha(G - v_i) = \alpha(G)$. By the α -minimality of G , $G - v_i$ is disconnected, and so $N_G(x) \cap V(G_i) = \{v_i\}$. Therefore the claim holds. \square

Claim 2.3. $|V(G_i)| = 2\alpha(G_i)$ for every $i \in \{2, \dots, k\}$.

Proof. Suppose that $|V(G_i)| \geq 2\alpha(G_i)$ for some $i \in \{2, \dots, k\}$. By Claim 2.2, there is a vertex $v_i \in V(G_i)$ such that $N_G(x) \cap V(G_i) = \{v_i\}$, $\alpha(G_i - v_i) = \alpha(G_i)$, and $\alpha(G - v_i) = \alpha(G)$. We will show that $G'_i = G_i - v_i$ is α -minimal. Suppose that there is a vertex y in $V(G'_i)$ such that $G'_i - y$ is connected and $\alpha(G'_i) = \alpha(G'_i - y)$. By Claim 2.2, v_i is a cut-vertex of G such that $G - v_i$ has exactly two components $G - V(G_i)$ and $G_i - v_i$. Then we have

$$\alpha(G'_i - y) + \alpha(G - V(G_i)) = \alpha(G - v_i - y) \leq \alpha(G - y) \leq \alpha(G) = \alpha(G - v_i) = \alpha(G'_i) + \alpha(G - V(G_i)),$$

which implies that $\alpha(G - v_i - y) = \alpha(G - y) = \alpha(G)$. Thus $G - y$ is disconnected by the α -minimality of G and so $N_G(v_i) = \{x, y\}$. For a maximum independent set I' of $G - v_i - y$ with size $\alpha(G)$, since v is a pendent vertex of $G - v_i - y$, we may assume that $v \in I'$ and so $x \notin I'$, which implies that $I' \cup \{v_i\}$ is an independent set of G whose size is greater than $\alpha(G)$, a contradiction. Therefore, G'_i is α -minimal.

By the induction hypothesis, $|V(G_i)| - 1 = |V(G'_i)| \leq 2\alpha(G'_i) - 1 = 2\alpha(G_i) - 1$, and thus $|V(G_i)| \leq 2\alpha(G_i)$. Hence, we can conclude that $|V(G_i)| = 2\alpha(G_i)$ for every $i \in \{2, \dots, k\}$. Then

$$2\alpha(G) - 1 \leq |V(G)| - 1 = \sum_{i \in [k]} |V(G_i)| \leq 1 + \sum_{i=2}^k 2\alpha(G_i) = 2\alpha(G) - 1,$$

which implies that $|V(G_i)| = 2\alpha(G_i)$ for every $i \in \{2, \dots, k\}$. \square

By Claims 2.2 and 2.3, for each $i \in \{2, \dots, k\}$, there is a vertex $v_i \in V(G_i)$ such that $N_G(x) \cap V(G_i) = \{v_i\}$ and $\alpha(G_i - v_i) = \alpha(G_i)$. We take a maximum independent set I_i of $G_i - v_i$ for each $i \in \{2, \dots, k\}$. Then $I' = (\cup_{i=2}^k I_i) \cup \{x\}$ is an independent set of $G - v$. Note that $|I'| = 1 + \sum_{i=2}^k \alpha(G_i - v_i) = \sum_{i \in [k]} \alpha(G_i) = \alpha(G)$. Thus I' is a maximum independent set of G not containing the vertex v , which is a contradiction. \square

Remark 2.4. As a corollary of Lemma 2.1, we have $\text{pc}_{\text{opt}}(G) \leq 3\alpha(G) - 2$. To see this, by deleting vertices of G one by one as long as the resulting graph is connected and has independence number $\alpha(G)$, we find an induced connected subgraph H of G such that $\alpha(H) = \alpha(G)$ and H is α -minimal. Take a spanning tree T of H and then recolor all the edges of T with $\Delta(T)$ new colors so that T is properly connected. Note that this recoloring is possible since every bipartite graph with maximum degree d has a proper edge coloring with d colors (see [1]). From the fact that $\alpha(G[V(T)]) = \alpha(G)$, every vertex in $V(G) \setminus V(T)$ has a neighbor in $V(T)$ and for every two nonadjacent vertices x, y in $V(G) \setminus V(T)$, there exist two distinct vertices $x', y' \in V(T)$ such that $xx', yy' \in E(G)$. Thus together with edges of color 1 between $V(G) \setminus V(T)$ and $V(T)$, one can check that G is properly connected. Hence,

$$\text{pc}_{\text{opt}}(G) \leq \Delta(T) + |E(T)| \leq \alpha(G) + (2\alpha(G) - 2) = 3\alpha(G) - 2.$$

It is not enough to obtain Theorem 1.2 or Theorem 1.3, but the idea of this argument plays a key role in their proofs.

Lemma 2.5. *For a connected graph G , there is a connected induced subgraph H of G such that $\alpha(H) = \alpha(G)$ and $|V(H)| \leq 2\alpha(G) - 1$. If such H has minimum number of vertices, then for every spanning tree T of H , every pendent vertex of T belongs to the same maximum independent set of H and so T has at most $\alpha(G)$ pendent vertices.*

Proof. By deleting vertices of G one by one as long as the resulting graph is connected and has independence number $\alpha(G)$, we can find a connected induced subgraph H of G such that $\alpha(H) = \alpha(G)$. By Lemma 2.1, $|V(H)| \leq 2\alpha(H) - 1 = 2\alpha(G) - 1$.

Take such induced subgraph H with minimum number of vertices. Take a maximum independent set I of G such that $I \subset V(H)$. If a spanning tree T of H has a pendent vertex x not in I , then $H - x$ is a connected induced subgraph of G such that $\alpha(H - x) = \alpha(G)$, which contradicts the minimality of $|V(H)|$. Thus for every spanning tree T of H , every pendent vertex of T belongs to I , and so T has at most $\alpha(G)$ pendent vertices. \square

Lemma 2.6. *Let T be a tree with at least two vertices and at most p pendent vertices. If $\Delta(T) > \frac{p+3}{2}$, then there exists a unique vertex of maximum degree and any vertex with second largest degree has degree at most $\Delta(T) - 2$.*

Proof. For simplicity, let $d = \Delta(T)$. Suppose that $d > \frac{p+3}{2}$. Note that since $p \geq 2$, $d \geq 3$. Let v be a vertex of maximum degree d , and w be a vertex with largest degree among vertices in $V(T) \setminus \{v\}$. Suppose to the contrary that $\deg_T(w) \geq d - 1$. Let e be the first edge on the path $P_T(v, w)$. Let T_v (resp. T_w) be the component of $T - e$ containing v (resp. w). Then $\Delta(T_v) \geq d - 1 (\geq 2)$ and so T_v has at least $d - 1$ pendent vertices and those are also pendent vertices of T . If $e \neq vw$, then $\Delta(T_w) = \deg_T(w)$ and so T_w has at least $d - 1$ pendent vertices and those except one are also pendent vertices of T . If $e = vw$, then T_w has at least $d - 2$ pendent vertices and those are also pendent vertices of T . Hence, there are at least $(d - 1) + (d - 2)$ pendent vertices in G . Thus $2d - 3 \leq p$ and $d \leq \frac{p+3}{2}$, a contradiction. \square

Lemma 2.7. *Let T be a spanning tree of a connected graph H such that $\Delta(T)$ is as small as possible. If T has a unique vertex v of maximum degree and the second largest degree of T is at most $\Delta(T) - 2$, then $H - v$ has $\Delta(T)$ components.*

Proof. Take a component H' of $H - v$. It is enough to show that $|E_T(v, V(H'))| = 1$. It is also trivial that $|E_T(v, V(H'))| \geq 1$ since T is a spanning tree of H . Suppose to the contrary that $|E_T(v, V(H'))| \geq 2$. Let $E_T(v, V(H')) = \{vx_1, \dots, vx_k\}$, and C_i be the component of $T - v$ containing the vertex x_i for every $i \in [k]$. Since H' is connected, there are distinct i, j such that $E_{H'}(V(C_i), V(C_j)) \neq \emptyset$. Then $T' = (T - vx_i) + e$ for some $e \in E_{H'}(V(C_i), V(C_j))$ is also a spanning tree of H . Since the second largest degree of T is at most $\Delta(T) - 2$, we have $\Delta(T') \leq \Delta(T) - 1$, which is a contradiction to the minimality of $\Delta(T)$. \square

The following lemma contains a key idea for the proof of Theorem 1.2.

Lemma 2.8. *Let G be a connected graph with $\alpha(G) \geq 3$, and H be a connected induced subgraph of G with a minimum number of vertices such that $\alpha(H) = \alpha(G)$. For every spanning tree T of H ,*

$$\text{pc}_{\text{opt}}(G) \leq |E(T)| + \Delta(T) - |M|,$$

where M is any set of edges $ww' \in E(T)$ such that $\deg_T(w) = 1$, $\deg_T(w') = 2$, and the following property (§) holds:

(§) For every $x \in V(G) \setminus V(T)$, if $xw \in E(G)$ then $xw' \in E(G)$.

Proof. Let I be a maximum independent set of H , and hence of G . We color all edges in $E(T) - M$ with the new colors $2, 3, \dots, \Delta(T) + 1$ (this is possible since the edge chromatic number of a tree T is $\Delta(T)$). Since the number of new colors is $\Delta(T)$ and the number of edges colored by new colors is $|E(T)| - |M|$, it is enough to show that G is properly connected.

Let W be the set of pendent vertices of T incident to an edge of M . Note that $W \subset I$ by Lemma 2.5. For every $u \in V(G) \setminus V(H)$, we define

$$X(u) = (N_G(u) \cap I) - W \quad \text{and} \quad X'(u) = (N_G(u) \cap V(H)) - W.$$

By definition, it is clear that $X(u) \subset X'(u) \subset V(H)$ for every $u \in V(G) \setminus V(H)$.

Claim 2.9. For every $u \in V(G) \setminus V(H)$, $X'(u) \neq \emptyset$.

Proof. Take $u \in V(G) \setminus V(H)$. It is trivial that if $X(u) \neq \emptyset$ then $X'(u) \neq \emptyset$. Suppose that $X(u) = \emptyset$. Since $u \notin I$, $N_G(u) \cap I \neq \emptyset$ by the maximality of $|I|$ in G . Thus by the definition of $X(u)$, $(N_G(u) \cap I) \cap W \neq \emptyset$. Take $w \in (N_G(u) \cap I) \cap W$. Then $uw \in M$ for some vertex $w' \in V(H)$. By the property (§), $X'(u)$ contains w' and therefore $X'(u) \neq \emptyset$. \square

If xy and xz are edges of M for some three vertices x, y, z , then from the definition of M , it follows that $\deg_T(x) = 2$ and $\deg_T(y) = \deg_T(z) = 1$, which means that T is a path with three vertices, a contradiction with the fact that $\alpha(H) = \alpha(G) \geq 3$. Hence, M is a matching.

Now we take two vertices u, v of G that are not adjacent. If $u, v \in V(H)$, then we can easily check that every path $P_T(u, v)$ is properly colored thanks to the fact that M is a matching.

Suppose that $u \notin V(H)$ and $v \in V(H)$. If u has a neighbor u' such that $u' \in V(H)$ and the first edge of the path $P_T(u', v)$ does not use the color 1, then $u + P_T(u', v)$ gives a properly colored path from u to v . Suppose to the contrary that for every vertex $u' \in N_G(u) \cap V(H)$, the first edge of the path $P_T(u', v)$ uses the color 1. We can take a vertex $u' \in X'(u)$ by Claim 2.9. Then the first edge of $P_T(u', v)$ uses color 1 and so lies in M . Then $u' \notin W$, $u'v \in M$ and therefore $v \in W$. Recall that $v \in W \subset I$, and so $u' \notin I$. By the maximality of $|I|$ in G , u has a neighbor z in $I \setminus \{u'\}$. By our case assumption that the first edge of $P_T(z, v)$ uses color 1, we have $z \in W$ and $zz' \in M$ for some $z' \in V(T)$. By the property (§), $uz' \in E(G)$. If $z' = u'$ then M cannot be a matching, a contradiction. Thus $z' \neq u'$, and so the first edge of $P_T(z', v)$ uses a color other than 1, which is a contradiction.

Now suppose that $u, v \notin V(H)$. If we can take $u' \in X'(u)$ and $v' \in X'(v)$ so that $u' \neq v'$, then $u + P_T(u', v') + v$ gives a properly colored path from u to v . Suppose to the contrary that we cannot take distinct $u' \in X'(u)$ and $v' \in X'(v)$. By Claim 2.9, neither $X'(u)$ nor $X'(v)$ is an emptyset. Thus $X'(u) = X'(v) = \{z\}$ for some vertex $z \in V(H) \setminus W$. By the maximality of $|I|$ in G , $(I \setminus \{z\}) \cup \{u, v\}$ is not an independent set in G , and therefore, u or v has a neighbor $w \in I \setminus \{z\}$. Since $X'(u) = X'(v) = \{z\}$, it follows that $w \in W$ and $ww' \in M$ for some vertex $w' \in V(H) \setminus W$. By the property (§), it holds that $w' \in X'(u) = X'(v) = \{z\}$ and so $z = w'$. Since $(I - \{w\}) \cup \{u, v\}$ is not an independent set of G by the maximality of $|I|$, u or v has a neighbor $y \in I \setminus \{w\}$. Then $y \in I \setminus \{w, w'\}$, since $w \in W \subset I$ and so $w' \notin I$. By definition, since $X'(u) = X'(v) = \{w'\}$, it holds that $y \in W$, and $yy' \in M$ for some vertex $y' \in V(H) \setminus W$. By the property (§), $y' \in X'(u) = X'(v) = \{w'\}$, and so $yy' = ww'$, which is a contradiction to the fact that $y \in I \setminus \{w, w'\}$. \square

3. Proof of Theorem 1.2

Proof of Theorem 1.2. Recall Theorem 1.1 and the fact that $\alpha(G) = 1$ if and only if $\text{pc}_{\text{opt}}(G) = 0$. Thus the theorem holds when $\alpha(G) \leq 2$. Let G be a connected graph with $\alpha(G) \geq 3$. We take a connected induced subgraph H of G with $\alpha(H) = \alpha(G)$, a spanning tree T of H , a maximum independent set I of H so that

- (1) $|V(H)|$ is minimum,
- (2) $\Delta(T)$ is minimum subject to (1), and
- (3) $|I \cap \{x \in V(H) \mid \deg_T(x) = \Delta(T)\}|$ is minimum subject to (2).

By Lemma 2.5 and by the choice of H , T and I , it is easy to see that the following holds:

- $|V(H)| \leq 2\alpha(G) - 1$,
- T is a tree with at most $\alpha(G)$ pendent vertices, and
- I is a maximum independent set of G .

If $\Delta(T) \leq \frac{\alpha(G)+3}{2}$, then by Lemma 2.8,

$$\text{pc}_{\text{opt}}(G) \leq |E(T)| + \Delta(T) \leq (2\alpha(G) - 2) + \frac{\alpha(G) + 3}{2} \leq \frac{5\alpha(G) - 1}{2}.$$

In the following, we suppose that $\Delta(T) > \frac{\alpha(G)+3}{2}$. By Lemma 2.6, there is a unique vertex v in T with maximum degree and so the condition (3) for the choice of I is equivalent to say ‘ $|I \cap \{v\}|$ is minimum’. We also note that by Lemma 2.6, the second largest degree of T is at most $\Delta(T) - 2$. We let $d = \Delta(T)$. By Lemma 2.7, $H - v$ has exactly d components. Let H_1, \dots, H_d be the components of $H - v$, and let $H_i^* = G[V(H_i) \cup \{v\}]$ for every $i \in [d]$.

Claim 3.1. *For every $i \in [d]$, $\alpha(H_i) = |I \cap V(H_i)|$. Moreover, if $v \in I$ then $\alpha(H_i^*) = |I \cap V(H_i^*)| = \alpha(H_i) + 1$ and every maximum independent set of H_i^* contains v .*

Proof. It is clear that $|I \cap V(H_i)| \leq \alpha(H_i)$ for every $i \in [d]$. Since the union of independent sets of H_i ’s is also an independent set of G , $\sum_{i \in [d]} \alpha(H_i) \leq \alpha(G)$. If $v \notin I$, then the claim holds clearly, since

$$\alpha(G) = |I| = \sum_{i \in [d]} |I \cap V(H_i)| \leq \sum_{i \in [d]} \alpha(H_i) \leq \alpha(G).$$

Suppose that $v \in I$. Then it holds that

$$\alpha(G) - 1 = |I| - 1 = \sum_{i \in [d]} |I \cap V(H_i)| \leq \sum_{i \in [d]} \alpha(H_i) \leq \alpha(G). \quad (3.1)$$

Suppose to the contrary that there is some $i \in [d]$ such that $\alpha(H_i) > |I \cap V(H_i)|$. Then from (3.1), $\alpha(H_i) = |I \cap V(H_i)| + 1 = |I \cap V(H_i^*)|$ and $\alpha(H_j) = |I \cap V(H_j)|$ for every $j \in [d]$ with $j \neq i$. Take a maximum independent set I'_i of H_i . Then $I' = I'_i \cup (I - V(H_i^*))$ is an independent set of G . Since $|I'| = |I'_i| + |I| - |I \cap V(H_i^*)| = |I|$, I' is a maximum independent set of G . Then $I' \subset V(H)$ and $v \notin I'$, which is a contradiction to the condition (3) for the choice of I . Hence, $\alpha(H_i) = |I \cap V(H_i)|$. This also implies that $\alpha(H_i) + 1 \geq \alpha(H_i^*) \geq |I \cap V(H_i^*)| = \alpha(H_i) + 1$. Thus $\alpha(H_i) + 1 = \alpha(H_i^*)$ and every maximum independent set of H_i^* contains v . \square

Claim 3.2. *For every $i \in [d]$, $|V(H_i)| \leq 2\alpha(H_i)$.*

Proof. Suppose to the contrary that $|V(H_i)| \geq 2\alpha(H_i) + 1$ for some $i \in [d]$. First, suppose that $v \notin I$. We take a connected induced subgraph H'_i of H_i with the minimum number of vertices such that $\alpha(H'_i) = \alpha(H_i)$. By Lemma 2.5, $|V(H'_i)| \leq 2\alpha(H_i) - 1$. It is enough to show that $|V(H_i) - V(H'_i)| \leq 1$. For simplicity, let $X = V(H_i) - V(H'_i)$. Suppose to the contrary that $|X| \geq 2$. If $V(H'_i)$ contains a neighbor of v then $H' = H - X$ is a connected induced subgraph of G such that $\alpha(H') = \alpha(G)$ and $|V(H')| < |V(H)|$, which is a contradiction to the condition (1) for the choice of H . Thus $V(H'_i)$ does not contain a neighbor of v .

Then there is an edge xv of G for some $x \in X$. Since H'_i contains a maximum independent set I' of H_i , $E_G(x, V(H'_i)) \neq \emptyset$ by the maximality of $|I'|$ in H_i . Then $H' := H - (X \setminus \{x\})$ is a connected induced subgraph of G such that $\alpha(H') = \alpha(G)$ and $|V(H')| < |V(H)|$, which is a contradiction to the condition (1) for the choice of H .

Secondly, suppose that $v \in I$. We take a minimal connected induced subgraph H'_i of H_i^* such that $\alpha(H'_i) = \alpha(H_i^*)$. By Lemma 2.5 and Claim 3.1, $|V(H'_i)| \leq 2(\alpha(H_i) + 1) - 1 = 2\alpha(H_i) + 1$. By Claim 3.1 again, v belongs to every maximum independent set of H_i^* and so $v \in V(H'_i)$. Consider $H' = H - X$ where $X = V(H_i^*) - V(H'_i)$. Then $\alpha(H') = \alpha(H'_i) + |I - (I \cap V(H_i^*))| = \alpha(H)$. If $X \neq \emptyset$ then $|V(H')| < |V(H)|$, which is a contradiction to the condition (1) for the choice of H . Thus $X = \emptyset$ and so $|V(H_i)| = |V(H'_i)| - 1 \leq 2\alpha(H_i)$. \square

Let $\mathcal{S}_2 = \{H_i \mid |V(H_i)| = 2\}$ and $\mathcal{S}_4 = \{H_i \mid |V(H_i)| \geq 4\}$. Then

$$1 + (d - |\mathcal{S}_4| - |\mathcal{S}_2|) + 2|\mathcal{S}_2| + 4|\mathcal{S}_4| \leq |V(H)| \leq 2\alpha(G) - 1,$$

and so $|\mathcal{S}_2| + 3|\mathcal{S}_4| \leq 2\alpha(G) - d - 2$. Then

$$|\mathcal{S}_4| \leq \frac{2\alpha(G) - d - 2 - |\mathcal{S}_2|}{3}. \quad (3.2)$$

Moreover, if $H_i \notin \mathcal{S}_2 \cup \mathcal{S}_4$, then $|V(H_i)| = 1$ or 3 and so $|V(H_i)| \leq 2\alpha(H_i) - 1$ by Claim 3.2. Hence, by Claim 3.2 again,

$$|V(T)| \leq 1 + \sum_{i \in [d]} (2\alpha(H_i) - 1) + |\mathcal{S}_2| + |\mathcal{S}_4| \leq 1 + 2\alpha(G) - d + |\mathcal{S}_2| + |\mathcal{S}_4|.$$

By applying Lemma 2.8,

$$\text{pc}_{\text{opt}}(G) \leq (2\alpha(G) - d + |\mathcal{S}_2| + |\mathcal{S}_4|) + d - |M| = 2\alpha(G) + |\mathcal{S}_2| + |\mathcal{S}_4| - |M|,$$

where M is any set of edges T in G satisfying the properties in Lemma 2.8. By (3.2),

$$\text{pc}_{\text{opt}}(G) \leq 2\alpha(G) + |\mathcal{S}_2| + \frac{2\alpha(G) - d - 2 - |\mathcal{S}_2|}{3} - |M| < \frac{5\alpha(G) - 1}{2} + \frac{2|\mathcal{S}_2| - 2}{3} - |M|,$$

where the last inequality is from the assumption that $d > \frac{\alpha(G)+3}{2}$.

Suppose that $|M| \geq |\mathcal{S}_2| - 1$. Then

$$\text{pc}_{\text{opt}}(G) \leq \frac{5\alpha(G) - 1}{2} + \frac{2|\mathcal{S}_2| - 2}{3} - (|\mathcal{S}_2| - 1) = \frac{5\alpha(G) - 1}{2} + \frac{-|\mathcal{S}_2| + 1}{3} \leq \frac{5\alpha(G) - 1}{2} + \frac{1}{3}.$$

Since $\text{pc}_{\text{opt}}(G)$ is an integer, $\text{pc}_{\text{opt}}(G) \leq \frac{5\alpha(G)-1}{2}$, and so the theorem holds. Thus, it is enough to show that $|M| \geq |\mathcal{S}_2| - 1$. Precisely, we will show that if $|\mathcal{S}_2| \geq 2$, then every edge of \mathcal{S}_2 satisfies the properties in Lemma 2.8, and thus we have $|M| \geq |\mathcal{S}_2|$ in this case.

Suppose that $|\mathcal{S}_2| \geq 2$. Since $\mathcal{S}_2 \neq \emptyset$, the minimality of $|V(H)|$ implies that $v \in I$. Take $H_i \in \mathcal{S}_2$. Let $V(H_i) = \{w, w'\}$, and without loss of generality, we may assume that $\deg_T(w) = 1$ and $\deg_T(w') = 2$. Suppose to the contrary that there is a vertex $x \in V(G) \setminus V(H)$ with $xw \in E(G)$ and $xw' \notin E(G)$. We first show that

$$N_G(x) \cap V(H) \subset \{v, w\}. \quad (3.3)$$

To show (3.3) by contradiction, suppose that there is $y \in N_G(x) \cap V(H_j)$ for some j with $j \neq i$. Consider the subgraph H' of G induced by $(V(H) - \{w'\}) \cup \{x\}$. Let T' be a tree obtained from $T - w'$ by adding

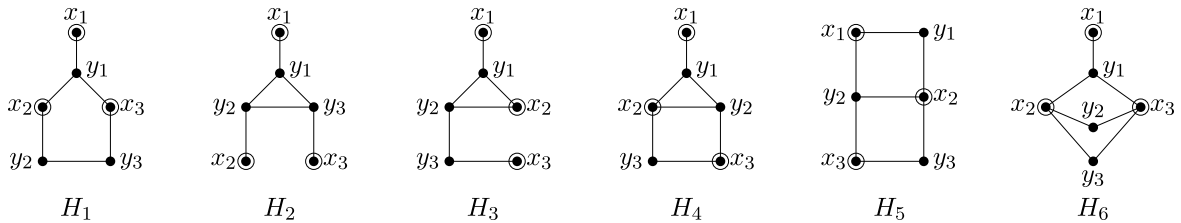


Fig. 4.1. Subgraphs H_1 - H_6 of G , where the circled vertices x_1, x_2, x_3 form an independent set of G .

the vertex x and adding edges xw, xy . Then H' has I as an independent set (and therefore $\alpha(H') = \alpha(G)$) and T' as a spanning tree. Since $|V(H')| = |V(H)|$ and $\Delta(T') \leq \Delta(T) - 1$ by Lemma 2.6, we reach a contradiction to the condition (2) for the choice of T . Thus, $(N_G(x) \cap V(H_j)) = \emptyset$ for every $j \in [d]$ with $j \neq i$, which also implies (3.3).

Since $|\mathcal{S}_2| \geq 2$, we can take some $H_{i'} \in \mathcal{S}_2$ where $i \neq i'$. Let $V(H_{i'}) = \{z, z'\}$, where $\deg_T(z) = 1$ and $\deg_T(z') = 2$. Consider the subgraph H' of G induced by $(V(H) - \{z\}) \cup \{x\}$. Let T' be a tree obtained from $T - z$ by adding the vertex x and adding the edge xw . Note that $|V(H)| = |V(H')|$ and T' is a spanning tree of H' . Then v is a unique vertex of maximum degree in T' and $\Delta(T') = \Delta(T)$. By (3.3), $I' = (I - \{w, z, v\}) \cup \{x, w', z'\}$ is a maximum independent set of G such that $I' \subset V(H')$. Since $v \notin I'$, we reach a contradiction with the condition (3) for the choice of I . \square

4. Proof of Theorem 1.3

Lemma 4.1. For a connected graph G with $\alpha(G) = 3$, $\text{pc}_{\text{opt}}(G) \leq 4$ if one of the following holds:

- (i) $V(G)$ can be partitioned into three cliques;
- (ii) $V(G)$ can be partitioned into four cliques and there is a matching M of size three such that those four disjoint cliques together with the edges in M is a connected spanning subgraph of G ;
- (iii) G has a 6-cycle $x_1, y_1, x_2, y_2, x_3, y_3, x_1$ such that each of $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ is an independent set of G ;
- (iv) G has one of H_1 - H_6 in Fig. 4.1 as a subgraph and $\{x_1, x_2, x_3\}$ is an independent set of G .

Proof. If (i) holds, that is, $V(G)$ is partitioned into three cliques X_1, X_2, X_3 , then we may assume that there are two edges $e \in E_G(X_1, X_2)$ and $e' \in E_G(X_1, X_3)$, and coloring e and e' with new colors 2 and 3, respectively, gives a properly colored path between every two vertices of G . Thus in this case, $\text{pc}_{\text{opt}}(G) \leq 2 + 2 = 4$. If (ii) holds, then coloring the edges in M by the new color 2 gives a properly colored path between every two vertices of G , and so $\text{pc}_{\text{opt}}(G) \leq 1 + 3 = 4$. If (iii) holds, then since $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ are maximum independent sets in G , every vertex in $V(G) \setminus \{x_1, x_2, x_3, y_1, y_2, y_3\}$ has a neighbor in $\{x_1, x_2, x_3\}$ and a neighbor in $\{y_1, y_2, y_3\}$, and so coloring three edges x_1y_1, x_2y_2, x_3y_3 by the new color 2 gives a properly colored path between every two vertices of G , and so $\text{pc}_{\text{opt}}(G) \leq 1 + 3 = 4$. Hence if (i), (ii), or (iii) holds, then $\text{pc}_{\text{opt}}(G) \leq 4$.

Now suppose to the contrary that (iv) holds and $\text{pc}_{\text{opt}}(G) > 4$. Then G has H as a subgraph which is isomorphic to H_i for some $i \in [6]$, and $I = \{x_1, x_2, x_3\}$ is an independent set of G . We label the vertices depicted in Fig. 4.1. We color three edges x_1y_1, x_2y_2, x_3y_3 by the new color 2. Since $\text{pc}_{\text{opt}}(G) > 4$, there are two nonadjacent vertices w and w' of G which cannot be connected by a properly colored path. Since there is a properly colored path between any pair of two vertices in H , we may assume without loss of generality that $w \notin V(H)$.

First, suppose that H is isomorphic to H_i for some $i \in [3]$. Since $w x_k \in E(G)$ for some $k \in [3]$ by the maximality of $|I|$, there is a properly colored path from w to every vertex of H starting with w, x_k, y_k . Thus $w' \notin V(H)$. Since $\alpha(G) = 3$, we may assume that $w x_k, w' x_j \in E(G)$ for two distinct $k, j \in \{1, 2, 3\}$. Then we can find a properly colored path starting with w, x_k, y_k and ending with y_j, x_j, w' . Consequently, there is a properly colored path connecting w and w' , a contradiction.

Suppose that H is isomorphic to H_4 . From two properly colored paths $x_1, y_1, x_2, y_2, x_3, y_3$ and $x_3, y_3, x_2, y_2, y_1, x_1$, since the vertex w is adjacent to some x_i , we know that w reaches any vertex of $V(H)$ by a properly colored path. Thus, $w' \notin V(H)$. Moreover, since $w w' \notin E(G)$ and $w, w' \notin I$, there are distinct i, j such that $w x_i, w' x_j$ are edges of G , since $|I| = \alpha(G) = 3$. If i or j is 1, then we can find from $x_1, y_1, y_2, x_2, y_3, x_3$ a properly colored path between w and w' . Thus, this asserts that neither w nor w' is adjacent to x_1 , and so $I' = \{x_1, w, w'\}$ is an independent set of G . Without loss of generality, we may assume that $w x_2, w' x_3 \in E(G)$. Then y_3 has a neighbor in I' by maximality of $|I'|$. If $y_3 w \in E(G)$, then w, y_3, x_3, w' is a properly colored path between w and w' . If $y_3 w' \in E(G)$, then $w, x_2, y_2, x_3, y_3, w'$ is a properly colored path between w and w' . If $y_3 x_1 \in E(G)$, then $w, x_2, y_2, y_1, x_1, y_3, x_3, w'$ is a properly colored path between w and w' . Thus, we reach a contradiction.

Suppose that H is isomorphic to H_5 . By the condition (iii) (the roles of y_2 and y_3 being exchanged in the cycle), $\{y_1, y_2, y_3\}$ is not an independent set of G . If $y_1 y_3 \in E(G)$ then G has a subgraph isomorphic to H_1 with independent set I ($x_2 y_2$ of H_5 plays the role of $x_1 y_1$ of H_1 , and vice-versa), and if $y_1 y_2 \in E(G)$ or $y_2 y_3 \in E(G)$ then G has a subgraph isomorphic to H_4 with independent set I , which implies that we can reach a similar contradiction.

Suppose that H is isomorphic to H_6 . If $y_2 y_3 \in E(G)$ then G has a subgraph isomorphic to H_1 with independent set I , and if $y_1 y_2 \in E(G)$ or $y_1 y_3 \in E(G)$ then G has a subgraph isomorphic to H_4 with independent set I , a contradiction. Thus $\{y_1, y_2, y_3\}$ is an independent set of G . From the fact that w has a neighbor in $\{x_1, x_2, x_3\}$ and a neighbor in $\{y_1, y_2, y_3\}$, we can find a properly colored path from w to every vertex of $V(H)$. More precisely, if $w x_1 \in E(G)$, then we use a part of the path $w, x_1, y_1, x_2, y_2, x_3, y_3$. When $w x_1 \notin E(G)$, we use either a part of the path x_2, y_2, x_3, y_3 or of the path x_3, y_3, x_2, y_2 in order to reach all the vertices of H except x_1 and y_1 , and a part of the path $y_3, x_3, y_2, x_2, y_1, x_1$ for the rest. Thus, $w' \notin V(H)$.

If $w x_1 \in E(G)$, then using some part of the path $w, x_1, y_1, x_2, y_2, x_3, y_3$ and edge $w' y_t$ for some $t \in [3]$, we can find a properly colored path between w and w' , a contradiction. Thus $w x_1 \notin E(G)$. Similarly, $w' x_1 \notin E(G)$. Then $I' = \{w, w', x_1\}$ is an independent set of G . We may assume that $w x_2, w' x_3 \in E(G)$. If $y_2 x_1, y_3 x_1 \in E(G)$, then $y_2, x_1, y_3, x_3, y_1, x_2$ is a 6-cycle satisfying (iii), a contradiction. Thus, $y_2 x_1 \notin E(G)$ or $y_3 x_1 \notin E(G)$. Since each of y_2 and y_3 has a neighbor in I' by maximality of $|I'|$, we reach a contradiction by one of the following four observations: (1) If $y_2 w \in E(G)$, then $w, y_2, x_2, y_3, x_3, w'$ is a properly colored path; (2) if $y_2 w' \in E(G)$, then w, x_2, y_2, w' is a properly colored path; (3) if $y_3 w \in E(G)$, then w, y_3, x_3, w' is a properly colored path; (4) if $y_3 w' \in E(G)$, then $w, x_2, y_2, x_3, y_3, w'$ is a properly colored path. \square

Proof of Theorem 1.3. Let G be a connected graph with $\alpha(G) = 3$. Suppose to the contrary that $\text{pc}_{\text{opt}}(G) > 4$. Then G does not satisfy any condition in Lemma 4.1. By Lemma 2.5, we can find one of T_1, T_2, T_3 of Fig. 4.2 as an induced subgraph. When G has an induced copy of T_i for some $i \in [3]$, we always follow the label in Fig. 4.2. For simplicity, we let $I = \{x_1, x_2, x_3\}$, which is an independent set of G , $Z = V(G) \setminus V(T_i)$, and $Z_k = \{w \in Z \mid N_G(w) \cap I = \{x_k\}\}$ for every $k \in [3]$. Clearly $Z \neq \emptyset$ (since $\text{pc}_{\text{opt}}(G) > 3$) and $Z'_k = Z_k \cup \{x_k\}$ is a clique for every $k \in [3]$ (since $\alpha(T_i) = \alpha(G) = 3$ for each i).

(Case 1) G contains T_1 or T_2 as an induced subgraph.

Suppose that G contains an induced copy of T_1 . Let us show that the vertex v dominates Z'_i for some $i \in [3]$. We remove from Z the vertices that are dominated by v , and if one of the Z_i 's is empty then we are done. Otherwise, we pick v as well as one vertex (among the remaining ones) inside each Z_i . By definition,

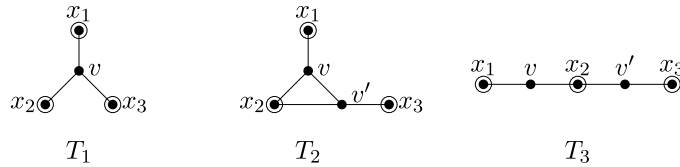


Fig. 4.2. Induced subgraphs T_1 - T_3 of G , where the circled vertices form an independent set of G .

it cannot yield an independent set (since $\alpha(G) = 3$), which necessarily means that G contains H_1 as a subgraph; a contradiction. Without loss of generality, we may assume that v dominates Z'_1 .

If $Z_1 \cup Z_2 \cup Z_3 = Z$, then $V(G)$ is partitioned into three cliques $Z'_1 \cup \{v\}$, Z'_2 , Z'_3 , which is the condition (i) of Lemma 4.1, a contradiction. Thus there is $w \in Z$ such that $wx_s, wx_t \in E(G)$ for some distinct $s, t \in [3]$. Since G has neither H_5 nor H_6 in Fig. 4.1 as a subgraph with independent set I , we can conclude that $Z - (Z_1 \cup Z_2 \cup Z_3) = \{w\}$. If $\{s, t\} = \{2, 3\}$, then coloring vx_2 and vx_3 by the new color 2 and 3, respectively, gives a properly colored path between every two vertices of G , a contradiction. Thus, by the symmetry of the roles of $\{1, 2\}$ and $\{1, 3\}$, we may assume that $\{s, t\} = \{1, 2\}$. If $Z_2 = \emptyset$, then $V(G)$ can be partitioned into three cliques $Z'_1 \cup \{v\}$, $\{x_2, w\}$, Z'_3 , a contradiction. Thus $Z_2 \neq \emptyset$. We take $w' \in Z_2$. Then the four sets $Z'_1 \cup \{v\}$, $\{w, x_2\}$, Z_2 , and Z'_3 form a clique partition of $V(G)$, and the edges x_1w , x_2w' , x_3v , form a matching satisfying the condition (ii) of Lemma 4.1, a contradiction.

Suppose that G does not contain an induced copy of T_1 . Then G contains T_2 as an induced subgraph. Suppose that x_2 has a neighbor $w \in Z$. If $wx_1, wx_3 \notin E(G)$, then G has H_2 in Fig. 4.1 with independent set $\{w, x_1, x_3\}$, a contradiction.

Thus we may assume without loss of generality that $wx_3 \in E(G)$. Then this asserts that G has H_4 as a subgraph with independent set I , a contradiction. If x_2 has no neighbor in Z , then coloring vx_1 and $v'x_3$ by the new color 2 gives a properly colored path between every two vertices of G , a contradiction.

(Case 2) G contains neither T_1 nor T_2 as an induced subgraph.

Then G contains T_3 as induced subgraph. If there is a common neighbor w of x_1 and x_3 , then by the condition (iii) of Lemma 4.1, we may assume that $vw \in E(G)$. Then it is a contradiction since either $\{x_1, w, x_2, x_3\}$ induces T_1 or $\{x_1, v, w, x_2, x_3\}$ induces T_2 . Thus there is no common neighbor of x_1 and x_3 . This implies that $V(G)$ can be partitioned into three sets Z'_1 , Z'_3 and $N_G[x_2]$. Clearly, $N_G[x_2]$ is not a clique since $vv' \notin E(G)$.

Claim 4.2. $N_G(x_2)$ can be partitioned into two cliques.

Proof. For simplicity, let $N = N_G(x_2) \setminus \{v, v'\}$. If $N = \emptyset$, then it is clear. Suppose that $N \neq \emptyset$. Let $A = \{w \in N \mid ww \in E(G)\}$ and $A' = \{w \in N \mid ww' \in E(G)\}$. Since G has no induced copy of T_1 (with x_2 playing the role of v), $N = A \cup A'$.

Suppose that $w \in A$. If $wx_3 \in E(G)$, then we have H_4 with independent set I , a contradiction. Thus $wx_3 \notin E(G)$. If $wx_1 \notin E(G)$, then we have H_3 with independent set $\{w, x_1, x_3\}$, a contradiction. Thus $wx_1 \in E(G)$. Then, since G has no H_4 as a subgraph, $ww' \notin E(G)$. Hence, for every vertex $w \in A$, we have $N_G(w) \cap V(T_3) = \{x_1, v, x_2\}$.

For $w, w' \in A$ with $w \neq w'$, if $ww' \notin E(G)$ then $\{w, w', v', x_2\}$ induces T_1 , a contradiction. Hence, $A \cup \{v\}$ is a clique. Similarly we can show that $A' \cup \{v'\}$ is a clique. This implies the claim. \square

Let $\{A, A'\}$ be a clique bipartition of $N_G(x_2)$ as in Claim 4.2, say $v \in A$ and $v' \in A'$.

Suppose that there are $z_1 \in Z_1$ and $z_3 \in Z_3$ such that $vz_1 \notin E(G)$ and $v'z_3 \notin E(G)$. If $z_1z_3 \in E(G)$, then a matching $\{x_1v, z_1z_3, x_3v'\}$ and a partition $\{Z'_1, A \cup \{x_2\}, A', Z'_3\}$ of $V(G)$ satisfy the condition (ii)

of Lemma 4.1, a contradiction. Thus $z_1 z_3 \notin E(G)$. Since $\{z_1, z_3, v, v'\}$ is not an independent set of G , we may assume without loss of generality that $v z_3 \in E(G)$ (since $v z_3$ and $v' z_1$ play symmetric roles). Since $\{z_3, x_1, x_2\}$ is an independent set of G by the definition of Z_3 , $\{x_1, x_2, v, z_3\}$ induces T_1 , which is a contradiction to the assumption of (Case 2). Hence, without loss of generality, we assume that v dominates Z_1 . Then the four sets $X_1 = Z'_1 \cup \{v\}$, $X_2 = A \cup \{x_2\} - \{v\}$, $X_3 = A'$, $X_4 = Z'_3$ form a clique partition of $V(G)$. If $|X_2| = 1$, that is $X_2 = \{x_2\}$, then $X_1, X_2 \cup X_3, X_4$ form a clique partition of $V(G)$, a contradiction in view of the condition (i) of Lemma 4.1. Thus $|X_2| \geq 2$, and we take a vertex $a \in X_2 \setminus \{x_2\}$.

Suppose that $|X_3| = 1$, that is $X_3 = \{v'\}$. If $Z_3 = \emptyset$, then $X_1, X_2, \{v', x_3\}$ form a clique partition of $V(G)$, a contradiction in view of the condition (i) of Lemma 4.1. Thus $Z_3 \neq \emptyset$. We take a vertex $b \in Z_3$. Then $X_1, X_2, \{v', x_3\}, Z_3$ form a clique partition of $V(G)$ and $\{va, x_2 v', x_3 b\}$ is a matching satisfying the condition (ii) of Lemma 4.1, a contradiction. Thus $|X_3| \geq 2$, and so we can take $a' \in X_3 \setminus \{v'\}$. Then together with the four cliques X_1, X_2, X_3, X_4 , $\{va, x_2 a', v' x_3\}$ is a matching satisfying the condition (ii) of Lemma 4.1, a contradiction. \square

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